Recall pointed spaces (X, x_0) & (Y, y_0) are <u>homotopy equivalent</u> if there is a <u>homotopy equivalence</u>

$$f:(X,x_0)\to(Y,y_0)$$

i.e a pointed map with a <u>homotopy inverse</u>

ie. a map $g:(Y,y_0)\to(X,x_0)$ such that

$$f\circ g\simeq 1_Y \quad\text{and}\quad g\circ f\simeq 1_X.$$

<u>Thm 58.7</u>: If $f:(X,x_0)\to(Y,y_0)$ is a homotopy equivalence then $\pi_1(f):\pi_1(X,x_0)\to\pi_1(Y,y_0)$ is an isomorphism.

Pf.

Choose a homotopy inverse $g:(Y,y_0)\to(X,x_0)$, & show $\pi_1(g)$ is an inverse to $\pi_1(f)$.

$$\pi_1(g)\circ\pi_1(f)=\pi_1(g\circ f)=\pi_1(1_X)=1_{\pi_1(X,x_0)}$$

$$\pi_1(f)\circ\pi_1(g)=\pi_1(f\circ g)=\pi_1(1_Y)=1_{\pi_1(Y,y_0)}$$

$\square$

Recall: We say (X, x_0) and (Y, y_0) have the same <u>homotopy type</u> if there's a homotopy equivalence between them. (In the next homework, you'll check that homotopy equivalence is an equivalence relationship.)

Example 1: consider these subspaces of $\mathbb{R}^2$:

$$| \qquad I$$

← remove this point + get 3 connected components

These aren't homeomorphic: $I - \{x\}$ can have 3 connected components, but $| - \{x\}$ can have at most 2 connected components.

But they are homotopy equivalent. We have an inclusion

$$| \xrightarrow{i} I$$

and a retraction

$$I \xrightarrow{r} |$$

that takes the serifs to single points. Notice that

$$r \circ i = 1_|$$

$$i \circ r \neq 1_I$$

but

$$i \circ r \simeq 1_I$$

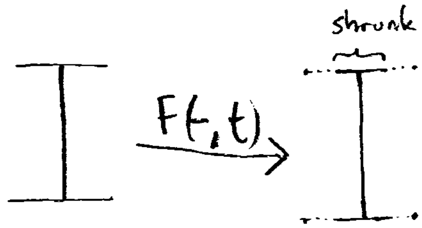
so i and r are homotopy equivalent.

There's a pointed homotopy

$$F: I \times [0,1] \to I$$

(where I is <u>not</u> the unit interval, but rather the seriffed subspace of $\mathbb{R}^2$) such that $F(x,0) = i \circ r$ & $F(x,1) = 1_I$:

for $t \in [0,1]$,

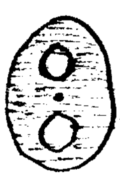


I.e. centering on the x_2 axis,

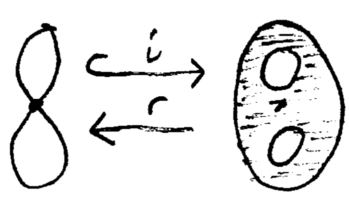
$$F((x_1,x_2),t) = (tx_1, x_2)$$

Example 2:

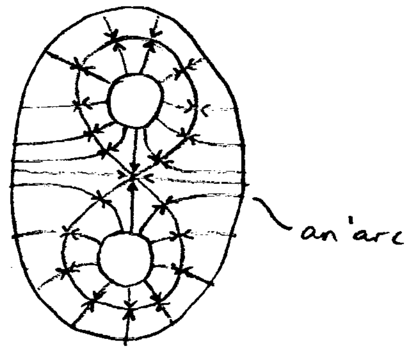
$X =$ 

$Y =$ 

Not homeomorphic, but homotopy equivalent. In a similar way to the last example, we have



Where $r \circ i$ maps Y to itself like this:



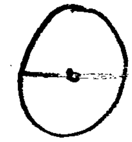
and the homotopy shrinks each arc by a factor of t.

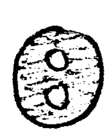
In both examples, our "small" space is a <u>deformation retract</u> of the big one, i.e. a subspace

$$(A, y_0) \subset (Y, y_0)$$

for which there exists a retraction $r : (Y, y_0) \to (A, y_0)$ such that $i \circ r \simeq 1_Y$ (a retraction automatically has $r \circ i = 1_A$).

So "A deformation retract of Y" $\Rightarrow$ "A homotopy equivalent to Y" but the converse isn't true:

Example 3:  is homotopy equivalent to 

but neither is a deformation retract of the other.
(Both are deformation retracts of .)