Homework

7 Feb 07

58.3 - Pointed version!

Hint: show if $f:(X,x_0)\to(Y,y_0)$

$g:(Y,y_0)\to(Z,z_0)$

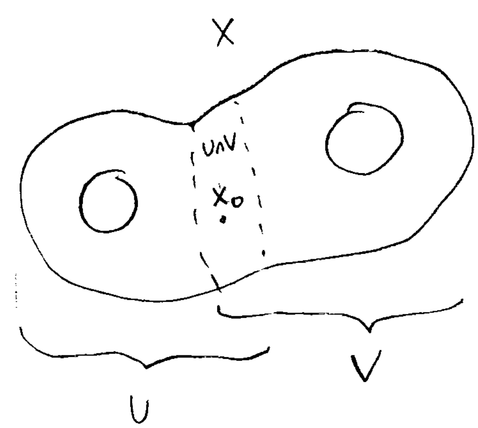
are pointed homotopy equivalences then so is $g\circ f$, etc.

58.6

59.3 Hint: remove a point x & use $\pi_1(\mathbb{R}^n - x, *)$

Seifert-van Kampen Thm, sections 59, 68-70

This lets us compute some fundamental groups of spaces that are the union of two open sets whose fundamental groups we already know.



From now on, assume:

- (X,x_0) pointed space
- U, V open in X
- $U\cup V = X$
- $x_0 \in U\cap V$
- $U\cap V$ connected

In this situation, we have inclusions of pointed spaces:

$$\begin{tikzcd} & (U\cap V, x_0) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ (U,x_0) \arrow[dr, "j_1"'] & & (V,x_0) \arrow[dl, "j_2"] \\ & (X,x_0) & \end{tikzcd}$$

This diagram commutes: $j_1 \circ i_1 = j_2 \circ i_2$.

Applying the functor π_1, we get a commuting diagram in Grp:

$$\begin{tikzcd}
 & \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\pi_1(U, x_0) \arrow[dr, "\pi_1(j_1)"'] & & \pi_1(V, x_0) \arrow[dl, "\pi_1(j_2)"] \\
 & \pi_1(X, x_0) &
\end{tikzcd}$$

~ not necessarily inclusion of groups!

The SvK theorem says what $\pi_1(X, x_0)$ is given everything else in the diagram. The answer is:

$\pi_1(X, x_0)$ is the <u>pushout</u> of

$$\begin{tikzcd}
 & \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\pi_1(U, x_0) & & \pi(V, x_0)
\end{tikzcd}$$

Where we'll define pushout later.

Recall if G is a group and $S \subset G$ is a subset then S <u>generates</u> G if every element of G can be written as a product of elements of S and their inverses. Equivalently, the smallest subgroup of G containing S is G itself.

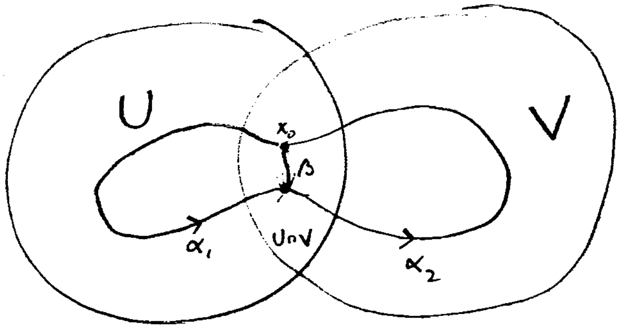
Baby S-vK Theorem (59.1): Given our assumptions,

$\pi_1(X, x_0)$ is generated by $\operatorname{im} \pi_1(j_1) \cup \operatorname{im} \pi_1(j_2)$

Pf.
We need to show that every element $[\gamma] \in \pi_1(X, x_0)$ is a product of elements of $\operatorname{im} \pi_1(j_1)$ and $\operatorname{im}(\pi_1 j_2)$, i.e.

$$[\gamma] = [\gamma_1] * [\gamma_2] * \cdots * [\gamma_n] \quad \text{where } \gamma_i \text{ is either in } \pi_1(U, x_0) \text{ or } \pi_1(V, x_0)$$

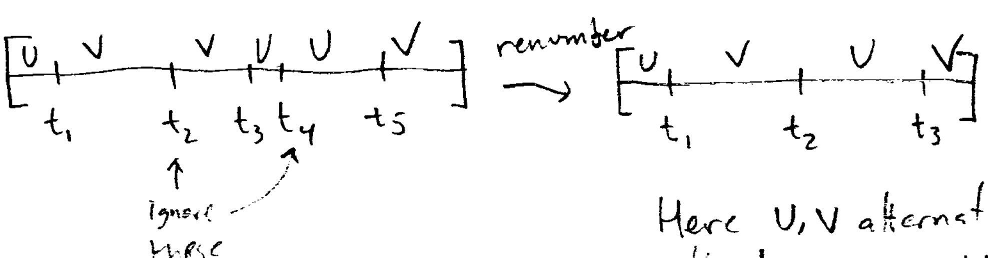
Here's a loop γ in $X = U \cup V$



$$[\gamma] = [\alpha_1] * [\alpha_2]$$
$$= \underbrace{[\alpha_1] * [\beta]}_{\text{loop in } U} * \underbrace{[\beta]^{-1} * [\alpha_2]}_{\text{loop in } V}$$

By Lebesgue number lemma, we can find $0 = t_0 < t_1 < \dots < t_n = 1$ st $\gamma[t_{i-1}, t_i]$ $(i = 1, \dots, n)$ lies either in U or in V

Without loss of generality, we can assume that $\gamma(t_i) \in U \cap V$:



Here U, V alternate so all t_i are in $U \cap V$