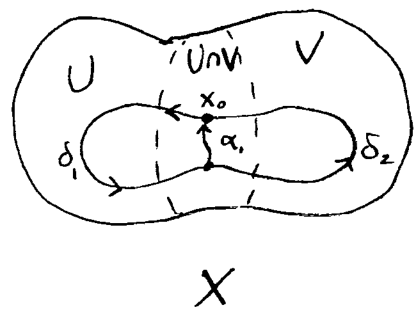
So, Assume we have $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\gamma[t_{i-1}, t_i]$ is contained in either U or V, with $\gamma(t_i)$ in $U \cap V$. Since $U \cap V$ is path connected, we can choose a path α_i from $\gamma(t_i)$ to x_0, lying in $U \cap V$. Now define paths δ_i to be the composites

$$[0,1] \xrightarrow[\text{the unique linear 1-1 onto map}]{} [t_{i-1}, t_i] \xrightarrow{\gamma|_{[t_{i-1}, t_i]}} X$$



Now we have

$$[\gamma] = \underbrace{[\delta_1] * [\alpha_1]} * \underbrace{[\alpha_1]^{-1} * [\delta_2] * [\alpha_2]} * \underbrace{[\alpha_2]^{-1} *} \cdots * \underbrace{[\alpha_{n-1}]^{-1} * [\delta_n]}$$

each a loop in either U or V

since δ_i are paths in U or V and α_i are paths in $U \cap V$.

Thus

$$[\gamma] = [\gamma_1] * \cdots * [\gamma_n]$$

where γ_i are loops in U or V.

$\square$

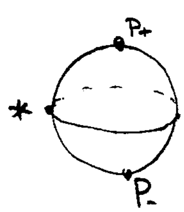
Cor. 59.2: Under the same assumptions, if U & V are simply connected, then X is simply connected.

Pf. Since U & V are simply connected, $\pi_1(U, x_0)$ & $\pi_1(V, x_0)$ are trivial groups, so $\operatorname{im} \pi_1(j_1) \cup \operatorname{im} \pi_1(j_2)$ is the trivial subgroup of $\pi_1(X, x_0)$. Baby S-vK says $\pi_1(X, x_0)$ is generated by this, so $\pi_1(X, x_0)$ is trivial and X is simply connected.

$$\begin{tikzcd} & U \cap V \arrow[dl, hook, "i_1"'] \arrow[dr, hook, "i_2"] & \\ U \arrow[dr, hook, "j_1"'] & & V \arrow[dl, hook', "j_2"] \\ & X & \end{tikzcd}$$

Thm 59.3 - If $n \geq 2$, the n-sphere is simply-connected, i.e. $\pi_1(S^n, *) \cong 1$.

Proof sketch: Use Baby S-vK with



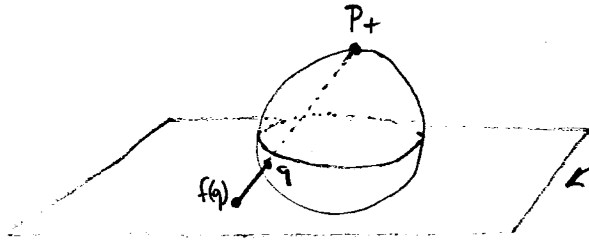
$P_+ = (0, 0, \ldots, 1) \in \mathbb{R}^{n+1}$
$P_- = (0, 0, \ldots, -1) \in \mathbb{R}^{n+1}$

Let $U = S^n - \{P_+\} \cong \mathbb{R}^n$ (homeomorphic)
$V = S^n - \{P_-\} \cong \mathbb{R}^n$
$U \cap V = S^n - \{P_+, P_-\} \cong S^{n-1} \times (-1, 1)$
(Note U, V simply connected and $U \cap V$ path connected if $n \geq 2$: S^0 is not connected!)

So Cor 59.2 implies the conclusion.

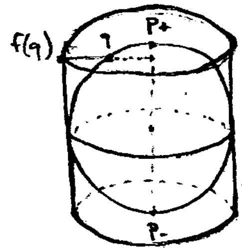
$\square$

There's a homeomorphism $f: S^n - \{p_+\} \to \mathbb{R}^n$ called <u>stereographic projection</u>:



the hyperplane $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ s.t. $x_{n+1} = -1\}$ $\cong \mathbb{R}^n$

There's a homeomorphism $g: S^n - \{p_+, p_-\} \to S^{n-1} \times (-1,1)$



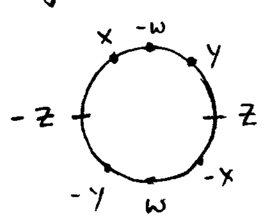
the cylinder $\{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$ st. $(x_1, \dots, x_n) \in S^n$ and $x_n \in (-1,1)\}$

Now let's look at a more interesting bunch of spaces:

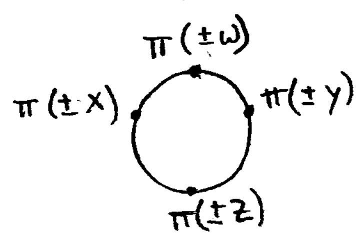
<u>Defn</u>: For any $n \geq 0$ we define <u>(real) projective n-space</u> $\mathbb{RP}^n$ to be the quotient space of S^n formed by identifying opposite points: $p \sim -p$ for all $p \in S^n$.

Consider the <u>projective line</u>:

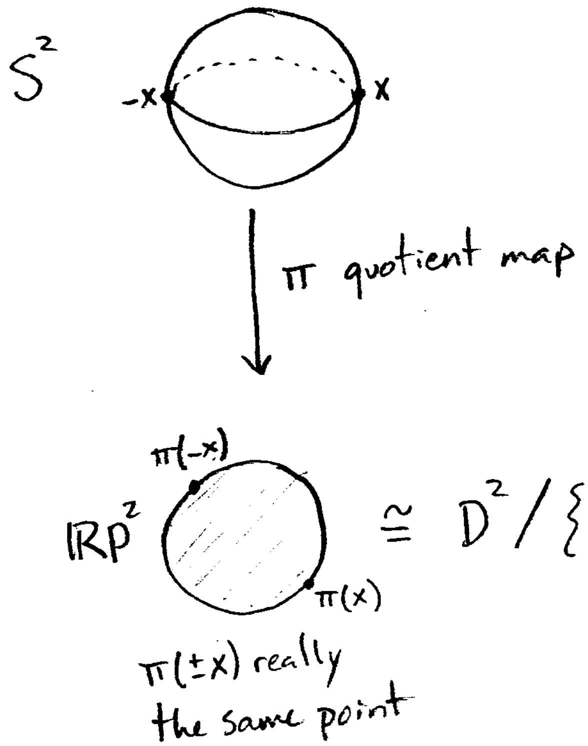
S^1



$\downarrow \pi$ quotient map is 2-1 and onto - actually a covering map



Next consider $\mathbb{R}P^2$, the <u>projective plane</u>:



$$\mathbb{R}P^2 \cong D^2 / \{ p \sim -p \text{ if } p \in S^1 \}$$