12 Feb '07

Recall $\mathbb{R}P^n$ is a quotient space of S^n

$$\mathbb{R}P^n = S^n / \{p \sim -p \text{ for } p \in S^n\}$$

i.e. $\mathbb{R}P^n$ is the set of equivalence classes of points in S^n where the equivalence relation is $p \sim q$ iff $q = -p$.

We have a function

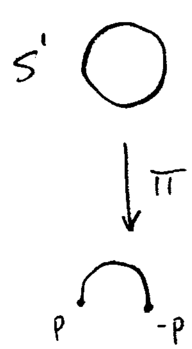
$$\begin{array}{c} S^n \\ \downarrow \pi \\ \mathbb{R}P^n \end{array}$$

with $\pi(p) = [p]$, i.e. p goes to its equivalence class.

We then make $\mathbb{R}P^n$ into a space where $U \subseteq \mathbb{R}P^n$ is open iff $\pi^{-1}(U)$ is open. Thus π is continuous; we call it the <u>quotient map</u>. All this works whenever we have any equivalence relation on any space.

We've seen

$n = 1$



Note $\mathbb{R}P^1 \cong D^1 / \{p \sim -p \text{ iff } p \in \delta D^1 = S^0\}$

$n=2$

$$S^2$$

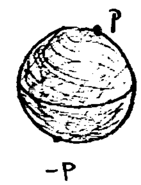
$$\big\downarrow \pi$$

$$\mathbb{R}P^2$$

$\mathbb{R}P^2 \cong D^2/\{p \sim -p \text{ iff } p \in \partial D^2 = S^1\}$

$n=3$

$$\begin{array}{c} S^3 \\ \big\downarrow \\ \mathbb{R}P^3 \cong D^3/\{p\sim -p \text{ iff } p \in \partial D^3 = S^2\} \end{array}$$



D^3

Thm 60.3 - If $n \geq 2$, $\pi_1(\mathbb{R}P^n, *) \cong \mathbb{Z}_2$.

Proof Sketch: First, show

$$\begin{array}{c} S^n \\ \big\downarrow \pi \\ \mathbb{R}P^n \end{array}$$

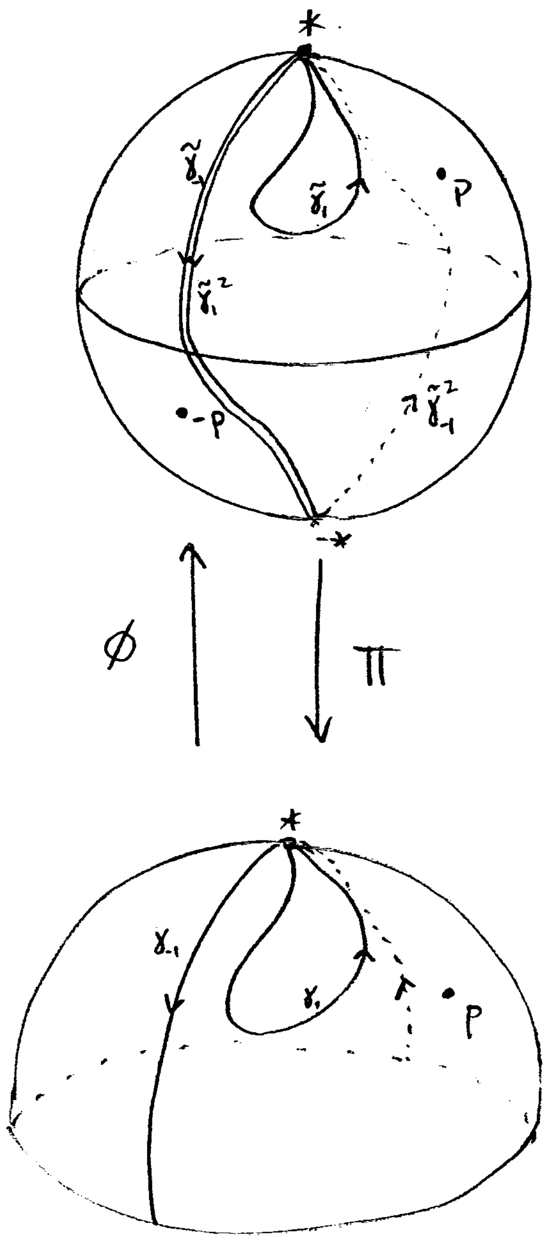
is a covering map, so we get a lifting map

$$\phi : \pi_1(\mathbb{R}P^n, *) \longrightarrow \pi^{-1}(*) \subseteq S^n.$$

In Thm 54.4 we saw that ϕ is a bijection if S^n is (path connected and) simply connected, which happens for $n \geq 2$ (thm 59.3). Since $\mathbb{R}P^n = S^n/\{p/\sim p \text{ for } p \in S^n\}$, $\pi^{-1}(*)$ is a 2-element set, so $\pi_1(\mathbb{R}P^n, *)$ is the unique 2-element group $\mathbb{Z}_2$.

$\square$

If $n=2$



Any loop γ_1 in $\mathbb{RP}^2$ has a unique lift to a path $\tilde{\gamma}_1$ in S^2 starting at $*$. If $\tilde{\gamma}_1$ ends at $*$ (i.e. if it's a loop) then we can contract $\tilde{\gamma}_1$ and therefore $\gamma_1 = \pi \circ \tilde{\gamma}_1$ so $[\gamma_1] = 1 \in \pi_1(\mathbb{RP}^2, *)$.

A more interesting case is when the path ends at $-*$, like it does for the loop $\gamma_{-1} \in \mathbb{RP}^2$. Traversing the loop twice lifts to a loop ending at $*$, so $[\gamma_{-1}] = -1 \in \pi_1(\mathbb{RP}^2, *)$.