Hw: 1) Show the quotient map $\pi : S^n \to \mathbb{R}P^n$ is a covering map (see Thm 60.3)

Also 60.2, 3.

<u>Seifert-van Kampen Thm</u>: Assume (X, x_0) is a pointed space, $U, V \subseteq X$ are open sets containing the basepoint x_0, $U \cup V = X$, and $U \cap V$ is path-connected. Then applying the functor $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$ to the commuting diagram

$$\begin{tikzcd} & (U\cap V, x_0) \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\ (U, x_0) \arrow[dr, hook, "j_1"'] & & (V, x_0) \arrow[dl, hook', "j_2"] \\ & (U\cup V, x_0) & \end{tikzcd}$$

we get a commuting diagram

$$\begin{tikzcd} & \pi_1(U\cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi(i_2)"] & \\ \pi_1(U, x_0) \arrow[dr, "\pi(j_1)"'] & & \pi_1(V, x_0) \arrow[dl, "\pi(j_2)"] \\ & \pi_1(U\cup V, x_0) & \end{tikzcd}$$

and this diagram is a pushout.

<u>Defn</u>: in any category, we say a commuting diagram

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "j_1"'] & & B_2 \arrow[dl, "j_2"] \\ & C & \end{tikzcd}$$

is a pushout if given any other commuting diagram

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "k_1"'] & & B_2 \arrow[dl, "k_2"] \\ & Q & \end{tikzcd}$$

the universal property

there exists a unique morphism $f: C \to Q$ such that

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "j_1"'] \arrow[ddr, bend right, "k_1"'] & & B_2 \arrow[dl, "j_2"] \arrow[ddl, bend left, "k_2"] \\ & C \arrow[d, dashed, "!f"] & \\ & Q & \end{tikzcd}$$

commutes. We often say C is the pushout of

$$\begin{tikzcd}[column sep=small] & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 & & B_2 \end{tikzcd}$$

but to be precise we need to specify j_1, j_2 as well.

Pushouts look scary, but in fact, they're just a generalization of the concept of the union of sets from the category Set to all other categories.

Here's how "union" is an example of a pushout:

Say U, V are two sets. Then we have a commuting diagram

$$\begin{tikzcd}
& U \cap V \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
U \arrow[dr, hook, "j_1"'] & & V \arrow[dl, hook', "j_2"] \\
& U \cup V &
\end{tikzcd}$$

in Set. We need to show that the universal property holds for $(U \cup V, j_1, j_2)$, i.e. given some other set Q and maps $k_1: U \to Q$, $k_2: V \to Q$ there exists a unique $f: U \cup V \to Q$ such that

$$\begin{tikzcd}
& U \cap V \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
U \arrow[dr, hook, "j_1"] \arrow[ddr, "k_1"'] & & V \arrow[dl, hook', "j_2"'] \arrow[ddl, "k_2"] \\
& U \cup V \arrow[d, dashed, "\exists! f"] & \\
& Q &
\end{tikzcd}$$

commutes,

Define
$$f \circ j_1 = k_1 ;$$
$$f \circ j_2 = k_2 .$$

To show f exists we have to show that f is well-defined on $U \cap V$. Say $x \in U \cap V$; then $i_1(x) = x \in U$, $i_2(x) = x \in V$. Our definition says $f \circ j_1 \circ i_1 = k_1 \circ i_1 \quad = \quad f \circ j_2 \circ i_2 = k_2 \circ i_2$ which is true because our diagram commutes.

The moral: to define a function $f : U \cup V \to Q$ we can give two functions $k_1 : U \to Q$, $k_2 : V \to Q$ that agree on $U \cap V$.

Another moral: if you can define a concept without referring to the underlying category, you can apply it anywhere.

Pushout example in Grp: Given

$$\begin{tikzcd}
 & 1 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
\mathbb{Z} & & \mathbb{Z}
\end{tikzcd}$$

where i_1, i_2 are the unique homomorphism from the trivial group 1 to $\mathbb{Z}$, the pushout is the free group on two generators

$$\begin{tikzcd}
 & & 1 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & & \\
 & \mathbb{Z} \arrow[ddr, "j_1"] & & \mathbb{Z} \arrow[ddl, "j_2"'] & \\
1 \arrow[drr, "a"'] & & & & 1 \arrow[dll, "b"] \\
 & & \mathbb{Z} * \mathbb{Z} & &
\end{tikzcd}$$