Homework

Given a space X and open sets $U, V \subseteq X$ and $x_0 \in U \cap V$, show

$$\begin{tikzcd} & (U\cap V, x_0) \arrow[dl, hook, "i_1"'] \arrow[dr, hook, "i_2"] & \\ (U, x_0) \arrow[dr, hook, "j_1"'] & & (V, x_0) \arrow[dl, hook, "j_2"] \\ & (U\cup V, x_0) & \end{tikzcd}$$

is a pushout in Top_*.

<u>Prop</u>: This diagram is a pushout in Grp:

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ \mathbb{Z} \arrow[dr, "j_1"'] & & \mathbb{Z} \arrow[dl, "j_2"] \\ & \mathbb{Z} * \mathbb{Z} = F_2 & \end{tikzcd}$$

Pf: Given any pair of homomorphisms $f_1, f_2 : \mathbb{Z} \to Q$, we need to check that $\exists!\, f : \mathbb{Z} * \mathbb{Z} \to Q$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
\mathbb{Z} \arrow[dr, "j_1"] \arrow[ddr, bend right, "f_1"'] & & \mathbb{Z} \arrow[dl, "j_2"'] \arrow[ddl, bend left, "f_2"] \\
 & \mathbb{Z} * \mathbb{Z} \arrow[d, dashed, "\exists ! f"] & \\
 & Q &
\end{tikzcd}
$$

Notice that

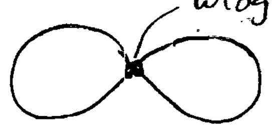
$$f(a_1) = f(j_1(1)) = f_1(1) \quad \text{and}$$
$$f(a_2) = f(j_2(1)) = f_2(1)$$

Since a_1, a_2 generate $\mathbb{Z} * \mathbb{Z}$, there's a unique f with these properties given f_1, f_2. f also exists, defined by these equations, since $\mathbb{Z} * \mathbb{Z}$ is free on a_1, a_2.

$\square$

Prop: If X is the figure-eight space & $x_0 \in X$, then $\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$

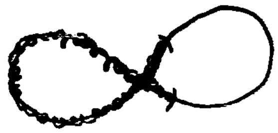
Pf.



wlog, we can take x_0 to be the point where the circles intersect.

We'll use the S-vK theorem.

Choose U, V as follows



U

V

Then $U \cap V$ is

Note that the assumptions of S-vK thm hold, so

$$\begin{tikzcd}
& \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\pi_1(U, x_0) \arrow[dr, "\pi_1(j_1)"'] & & \pi_1(V, x_0) \arrow[dl, "\pi_1(j_2)"] \\
& \begin{array}{c}\pi_1(U \cap V, x_0) \\ = \pi_1(X, x_0)\end{array} &
\end{tikzcd}$$

This diagram is a pushout.

We have

$$\pi_1(U \cap V) \cong 1$$

since it's contractible,

$$\pi_1(V,x_0) \cong \pi_1(U,x_0) \cong \pi_1(S^1,*) \cong \mathbb{Z}$$

since there are deformation retracts of U, V to S^1, so

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
\mathbb{Z} \arrow[dr, "\pi_1(j_1)"'] & & \mathbb{Z} \arrow[dl, "\pi_1(j_2)"] \\
 & \pi_1(X,x_0) &
\end{tikzcd}$$

is a pushout.

We know from the previous proposition that

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
\mathbb{Z} \arrow[dr] & & \mathbb{Z} \arrow[dl] \\
 & \mathbb{Z} * \mathbb{Z} &
\end{tikzcd}$$

is a pushout, so we can conclude

$$\pi_1(X,x_0) \cong \mathbb{Z} * \mathbb{Z}$$

by using the following lemma.

Lem: In any category, given two pushouts

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "j_1"'] & & B_2 \arrow[dl, "j_2"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "j_1'"'] & & B_2 \arrow[dl, "j_2'"] \\ & C' & \end{tikzcd}$$

we have $C \cong C'$.

Pf. This diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ B_1 \arrow[dr, "j_1"] \arrow[ddr, "j_1'"'] & & B_2 \arrow[dl, "j_2"'] \arrow[ddl, "j_2'"] \\ & C \arrow[d, dashed, shift right, "\exists! f"'] & \\ & C' \arrow[u, dashed, shift right, "\exists! g"'] & \end{tikzcd}$$

since both C, C' satisfy the universal property of pushouts. But there's a unique map h making this diagram commute:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B_1 \arrow[dr, "j_1"] \arrow[ddr, "j_1"'] & & B_2 \arrow[dl, "j_1"'] \arrow[ddl] \\ & C \arrow[d, dashed, "\exists! h"] & \\ & C & \end{tikzcd}$$

Namely, $h = 1_C$. Since it's unique, $g \circ f : C \to C$ must also equal 1_C; similarly, $f \circ g = 1_{C'}$. Thus $C \cong C'$.

$\square$

<u>Lem</u>: The pushout of every diagram

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
G & & H
\end{tikzcd}$$

in Grp exists:

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
G \arrow[dr, "j_1"'] & & H \arrow[dl, "j_2"] \\
 & G * H &
\end{tikzcd}$$

where $G * H$ is the free product of G and H.

The elements of $G * H$ are equivalence classes of symbols

$$g_1 * h_1 * g_2 * h_2 * \cdots * g_n * h_n \qquad n \geq 0$$

where $g_i \in G$, $h_i \in H$. The equivalence relation has

$$g_1 * h_1 * \cdots * h_i * 1_G * h_{i+1} * \cdots * h_n$$
$$\sim g_1 * h_1 * \cdots * h_i h_{i+1} * \cdots * h_n$$

and similarly for $g_i * 1_H * g_{i+1}$.

The product is $(g_1 * \cdots * h_n) * (g_{n+1} * \cdots * h_m) = g_1 * \cdots * h_n * g_{n+1} * \cdots * h_m$.

Pf. sketch:

Check that $G*H$ is a group. Define

$$\begin{aligned} j_1 : G &\to G*H \\ g &\mapsto g * 1_H \end{aligned}$$

$$\begin{aligned} j_2 : H &\to G*H \\ h &\mapsto 1_G * h \end{aligned}$$

Now show our diamond is a pushout:

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ G \arrow[dr, "j_1"] \arrow[ddr, "f_1"'] & & H \arrow[dl, "j_2"'] \arrow[ddl, "f_2"] \\ & G*H \arrow[d, dashed, "\exists! f"] & \\ & Q & \end{tikzcd}$$

Check that the only f making this commute is

$$f(g_1 * h_1 * \cdots * g_n * h_n) = f_1(g_1) f_2(h_1) \cdots f_1(g_n) f_2(h_n)$$

$\square$