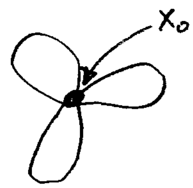
21 Feb '07

Last time we saw that given two groups G & H there's a pushout

$$\begin{tikzcd} & 1 \arrow[dl, hook] \arrow[dr, hook] & \\ G \arrow[dr, "j_1"'] & & H \arrow[dl, "j_2"] \\ & G*H & \end{tikzcd}$$

where $G*H$ is the free product of G & H, generated by elts. $g*1$ & $1*h$ with some obvious relations.

HW 1. Using this + S-vK Thm, show that π_1 of this pointed space X



"The boquet of three circles"

is $\mathbb{Z}*(\mathbb{Z}*\mathbb{Z})$. Then define "the boquet of n circles" & show (inductively using S-vK) that π_1 of this is $\underbrace{\mathbb{Z}*(\mathbb{Z}*(\dots(\mathbb{Z}*\mathbb{Z})\dots))}_{n \text{ copies of } \mathbb{Z}}$ and show this group is free on n generators.

Thm: For any group homomorphisms

$$\begin{tikzcd} & K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ G & & H \end{tikzcd}$$

There's a pushout

$$\begin{tikzcd} & K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ G \arrow[dr, "j_1"'] & & H \arrow[dl, "j_2"] \\ & G*_K H & \end{tikzcd}$$

(beware: $G*_K H$ also depends on i_1, i_2!) where

$$G*_K H = G*H/N$$

where N is the <u>normal</u> subgroup generated by all elts. of the form

$$(i_1(k)*1)(1*i_2(k))^{-1}.$$

Recall that the "normal subgroup generated by $\{x_i\}$" is the smallest normal subgroup containing all $\{x_i\}$, or equivalently, the subgroup formed by multiplying, inverting, + conjugating the $\{x_i\}$ ad infinitum.

Pf. Define $j_1 : G \longrightarrow G*_K H$, $g \longmapsto [g*1]$, $j_2 : H \longrightarrow G*_K H$, $h \longmapsto [1*h]$

To check that this gives a pushout, first check that our diamond commutes:

$$j_1(i_1(k)) = [i_1(k)*1] = [1*i_2(k)] = j_2(i_2(k))$$

since the ratio is in the normal subgroup

Next we need to check the universal property: given any commutative diamond

$$\begin{tikzcd}
& K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
G \arrow[dr, "q_1"'] & & H \arrow[dl, "q_2"] \\
& Q &
\end{tikzcd}$$

we get a unique morphism $f: G *_K H \to Q$ s.t. this diagram commutes:

$$\begin{tikzcd}
& K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
G \arrow[dr, "j_1"] \arrow[ddr, bend right, "q_1"'] & & H \arrow[dl, "j_2"'] \arrow[ddl, bend left, "q_2"] \\
& G *_K H \arrow[d, dashed, "f"] & \\
& Q &
\end{tikzcd}$$

To do this, first check that f is unique.

$$f[g*1] = q_1(g)$$

$$f[1*h] = q_2(h)$$

Since $g*1$, $1*h$ generate $G*H$, $[g*1]$ and $[1*h]$ generate $G*_K H$, so f is uniquely determined.

Now check that f exists, i.e. is well-defined. Note that $G*H$ is itself a pushout:

$$\begin{tikzcd}
 & 1 \arrow[dl] \arrow[dr] & \\
G \arrow[dr, "g \mapsto g*1"'] \arrow[ddr, bend right, "q_1"'] & & H \arrow[dl, "1*h \mapsfrom h"] \arrow[ddl, bend left, "q_2"] \\
 & G*H \arrow[d, dashed, "\exists!\phi"] & \\
 & Q &
\end{tikzcd}$$

Given any homomorphisms $q_1 : G \to Q$, $q_2 : H \to Q$, $\exists!\, \phi : G*H \to Q$ s.t. the diagram commutes.

Since

$$\begin{aligned}
\phi\big((i_1(k)*1)(1*i_2(k))^{-1}\big) &= \phi(i_1(k)*1)\,\phi(1*i_2(k))^{-1} \\
&= q_1(i_1(k))\, q_2(i_2(k))^{-1} \\
&= 1
\end{aligned}$$

we know $N \subseteq \ker\phi$ and therefore $\exists f$ s.t. $f \circ [\,] = \phi$:

$$\begin{tikzcd}
 & K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
G \arrow[dr, "g \mapsto g*1"] \arrow[ddr, "j_1"'] \arrow[dddr, bend right, "q_1"'] & & H \arrow[dl, "1*h \mapsfrom h"'] \arrow[ddl, "j_2"] \arrow[dddl, bend left, "q_2"] \\
 & G*H \arrow[d, "{[\,]}"] & \\
 & G*_K H \arrow[d, "f"] & \\
 & Q &
\end{tikzcd}$$