8 Jan 07
Derek

Def. Suppose $f, f' : I \to X$ are two paths from x_0 to x_1 in X.
A (path) homotopy from f to f' is a map

$$F : I \times I \to X$$

such that

$$F(s,0) = f(s) \quad \text{and} \quad F(0,t) = x_0 \quad \text{for all } t.$$
$$F(s,1) = f'(s) \qquad\qquad F(1,t) = x_1$$

Note: We can define $F_t : I \to X$, $s \mapsto F(s,t)$; then for all t, F_t is a path from x_0 to x_1.

Facts (proved in 205A):

- path homotopic is an equivalence relation $\simeq_p$
- We can define a group operation on paths. Given a path $x_0 \xrightarrow{f} x_1$ and a path $x_1 \xrightarrow{g} x_2$, then $x_0 \xrightarrow{f*g} x_2$ is defined to be

$$(f*g)(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}.$$

Then if $f \simeq_p f'$ and $g \simeq_p g'$, we have $f*g \simeq_p f'*g'$.
This implies that we get a well-defined operation on equivalence classes

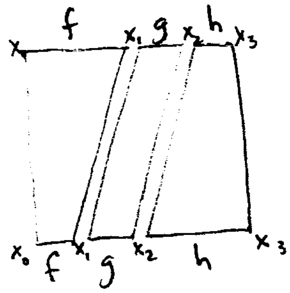
$$[f]*[g] := [f*g]$$

whenever $f*g$ is defined.

• If f, g, h paths in X $\quad x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2 \xrightarrow{h} x_3$

then $\quad [f*(g*h)] = [(f*g)*h]$.

Proof sketch:



Use this path homotopy

$\square$

Upshot: $\quad *$ is associative on homotopy classes.

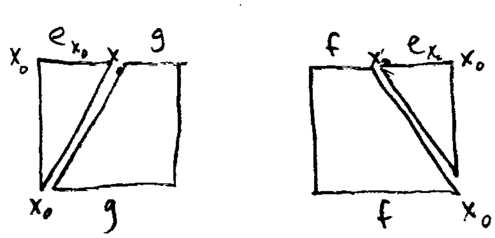
• If $e_{x_0} : I \to X$ is the constant path $e_{x_0}(s) = x_0$, then

$$[e_{x_0}] * [g] = [g]$$

$$[f] * [e_{x_0}] = [f]$$

whenever g starts at x_0 and f ends at x_0.

Proof sketch:

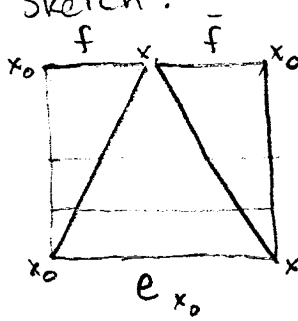


$\square$

- If $\bar{f}(s) = f(1-s)$ and $x_0 \xrightarrow{f} x_1$

$$[f]*[\bar{f}] = e_{x_0} \qquad [\bar{f}]*[f] = e_{x_1}$$

Proof sketch:



go all the way to x_1 then go back

go halfway along f, sit there for a while, then go back

go quarter way along f, sit there for longer, then go back.

sit there at x_0 the whole time

$\square$

Defn: A <u>pointed space</u> (X, x_0) is a space X with chosen point $x_0 \in X$ called the <u>basepoint</u>.

Defn: The <u>fundamental group</u> $\pi_1(X, x_0)$ of a pointed space (X, x_0) is the group with

- elements: (path) homotopy classes of loops based at x_0
- multiplication: concatenation $*$

Note: that this forms a group is a corollary of the Facts above:

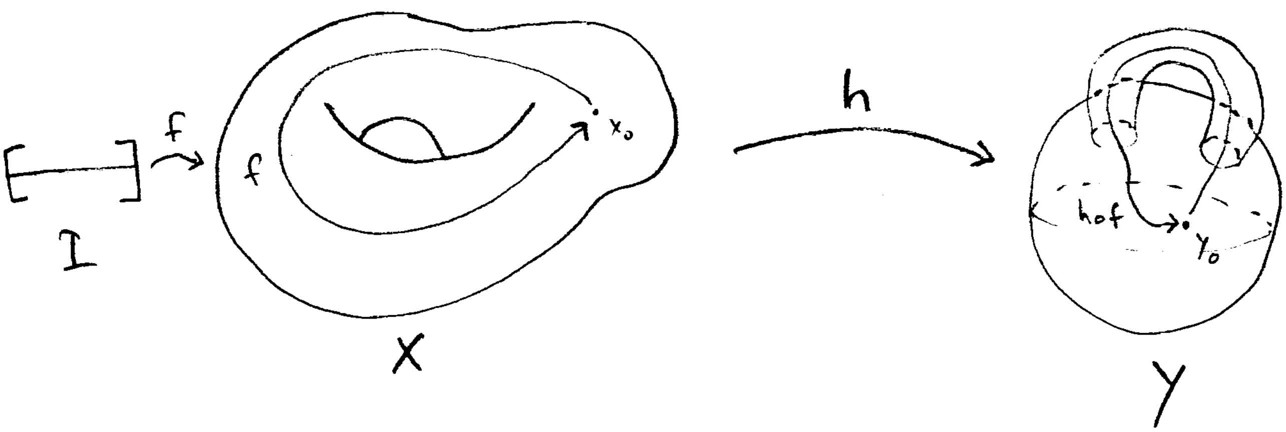
- $*$ always defined
- $*$ associative
- $\exists$ identity $[e_{x_0}]$
- $\forall [f]$, $[f]^{-1} = [\bar{f}]$

So we get a function

$$\pi_1 : \text{PointedSpaces} \longrightarrow \text{Groups}$$

Better yet, it's a <u>functor</u>, meaning it also does something nice for maps:

Given a map $h : (X, x_0) \to (Y, y_0)$, any loop $f : I \to (X, x_0)$ gives a loop $h \circ f : I \to (Y, y_0)$.



Note: If $f \simeq_p f'$ then $h \circ f \simeq_p h \circ f'$.

Pf. If F is a homotopy $f \xrightarrow{F} f'$
then $h \circ F$ is a homotopy $h \circ f \xrightarrow{h \circ F} h \circ f'$.

$\square$

Thus we get a well-defined map

$$\pi_1(h) : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[f] \longrightarrow [h \circ f]$$

Note: Munkres calls this h_*.

Thm. $\pi_1(h)$ is a homomorphism for any map h, i.e. $\pi_1(h)(f*g) = \pi_1(h)(f) * \pi_1(h)(g)$.
(Prove it yourself.)

Thm 52.4: π_1 is a functor, i.e. it respects composition + identities: given maps of pointed spaces

$$(X, x_0) \xrightarrow{h} (Y, y_0) \xrightarrow{k} (Z, z_0)$$

we have

$$\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$$

and if $1_{(X,x_0)}$ is the identity map, then

$$\pi_1\left(1_{(X,x_0)}\right) = 1_{\pi_1(X,x_0)}$$

Proof: Pick $[f] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}
\pi_1(k \circ h)([f]) &= [(k \circ h) \circ f] \\
&= [k \circ (h \circ f)] \\
&= \pi_1(k)([h \circ f]) \\
&= \pi_1(k) \circ \pi_1(h) \; ([f])
\end{aligned}$$

$$\pi_1\left(1_{(X,x_0)}\right)([f]) = [1_{(X,x_0)} \circ f] = [f] = 1_{\pi_1(X,x_0)}([f])$$

$\square$