Last time we showed that given any diagram in Grp like

$$\begin{tikzcd} & K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ G & & H \end{tikzcd}$$

we get a pushout

$$\begin{tikzcd} & K \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ G \arrow[dr, "j_1"'] & & H \arrow[dl, "j_2"] \\ & G *_K H & \end{tikzcd}$$

where $G *_K H = G*H/N$ and $N$ is the normal subgroup generated by $(i_1(k)*1)(1*i_2(k))^{-1}$ for all $k \in K$. Using this, we see

<u>Seifert-van Kampen Theorem</u>: If $(X,x_0)$ is a pointed space, $U$ and $V$ are open sets in $X$, $U \cap V$ is path-connected, $x_0 \in U \cap V$, and $U \cup V = X$ then

$$\pi_1(X,x_0) = \pi_1(U,x_0) * \pi_1(V,x_0)/N$$

where $N$ is the normal subgroup of $\pi_1(X,x_0)$ generated by

$$(\pi_1(i_1(k))*1)(1*\pi_1(i_2(k)))^{-1} \quad \forall\, k \in \pi_1(U \cap V, x_0)$$

where
$$i_1 : (U\cap V, x_0) \hookrightarrow (U, x_0)$$
$$i_2 : (U\cap V, x_0) \hookrightarrow (V, x_0)$$

Proof: use previous S-vK theorem + our description of the pushout of groups.

$\square$

Example 1: $\pi_1(T^2, *) \cong \mathbb{Z}^2$

We've seen this using the covering map $\mathbb{R}^2 \to T^2$.

Also,

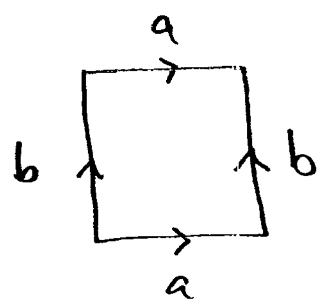
$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

(see the book), so

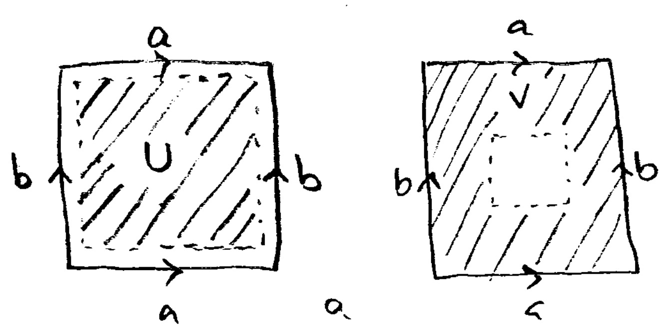
$$\pi_1(T^2, *) \cong \pi_1(S^1, *) \times \pi_1(S^1, *) \cong \mathbb{Z}^2$$

But let's compute it using S-vK.

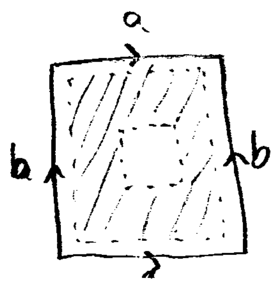
Note that $T^2$ is a quotient space of $I \times I$:




where we identify the edges as marked. Let $U, V \subseteq T^2$ be



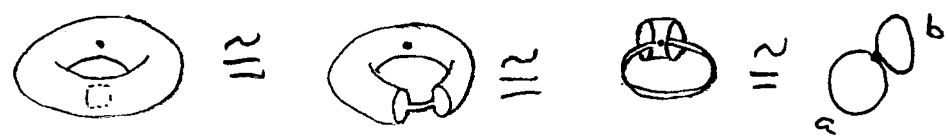

$U \cap V =$



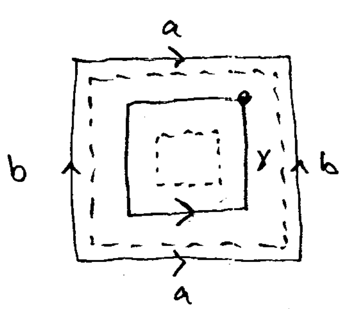
We get a pushout, by S-vK

$$\begin{tikzcd}
& \pi_1(U \cap V, x_0) \cong \mathbb{Z} \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
1 = \pi_1(U, x_0) \arrow[dr] & & \pi_1(V, x_0) \cong \mathbb{Z} * \mathbb{Z} \arrow[dl] \\
& \pi_1(T^2, x_0) &
\end{tikzcd}$$

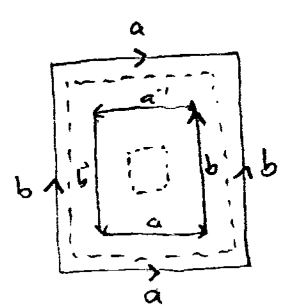
Since

1) $U \cap V$ has $S^1$ as a deformation retract
2) $U$ is contractible
3) $V \cong$ 


To describe $\pi_1(i_2) : \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$, we just need to compute $\pi_1(i_2(1))$, since $1$ generates $\mathbb{Z}$. Letting $[\gamma]$ be the generator of $\pi_1(U \cap V, x_0)$




we find that $\pi_1(i_2)([\gamma]) = a^{-1} b^{-1} a b$



So using the formula for a pushout of groups,

$$\pi_1(X, x_0) = \underline{1} * (\mathbb{Z} * \mathbb{Z})/N$$

where $N$ is as in the S-vK theorem. Note that

$$\underline{1} * G \cong G$$

(check it!) so $\pi_1(X, x_0)$ is the free group on $\{a, b\}$ mod the normal subgroup generated by

$$\{1,\ a^{-1}b^{-1}ab\}.$$

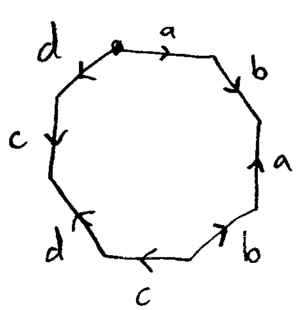
left side of pushout

right side

But this is just the free abelian group on $\{a, b\}$, i.e. $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.

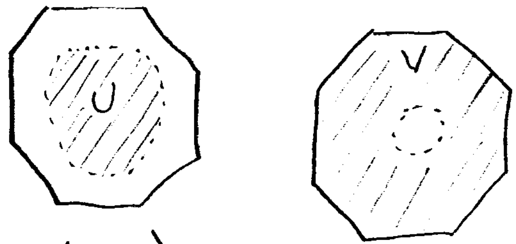
---

Homework 2: Show with a picture proof that the quotient of a solid octogon formed by identifying edges as below




is homeomorphic to the two-holed torus

and then use S-vK to compute $\pi_1$ of the space. (Hint: use




as your open sets.)

Homework #3. Let $X$ be the quotient space of a solid triangle formed as shown and compute $\pi_1(X,*)$.