26 Jan '07

Proof sketch of S-vK Thm (Theorem 70).

Let (X, x_0) be a pointed space w/ open sets $U, V \subseteq X$ such that $U \cup V = X$, $U \cap V$ contains x_0 and is path-connected. Then we have a pushout:

$$
\begin{tikzcd}
 & \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\pi_1(U, x_0) \arrow[dr, "\pi_1(j_1)"'] & & \pi_1(V, x_0) \arrow[dl, "\pi_1(j_2)"] \\
 & \pi_1(X, x_0) &
\end{tikzcd}
$$

where

$$
\begin{aligned}
i_1 &: (U \cap V, x_0) \hookrightarrow (U, x_0) \\
i_2 &: (U \cap V, x_0) \hookrightarrow (V, x_0) \\
j_1 &: (U, x_0) \hookrightarrow (X, x_0) \\
j_2 &: (V, x_0) \hookrightarrow (X, x_0)
\end{aligned}
$$

are inclusions.

Munkres proves only the special case where U, V are path-connected, so we'll sketch that case using his notation. (Read over the proof in Munkres) The general theorem reduces to this special case.

Pf. We need to show that the above diamond is a pushout, i.e. given any commutative diamond

$$\begin{tikzcd} & \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\ \pi_1(U, x_0) \arrow[dr, "\phi_1"'] & & \pi_1(V, x_0) \arrow[dl, "\phi_2"] \\ & H & \end{tikzcd}$$

there exists a unique (homo)morphism

$$\phi : \pi_1(X, x_0) \longrightarrow H$$

such that this diagram commutes:

$$\begin{tikzcd} & \pi_1(U \cap V, x_0) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\ \pi_1(U, x_0) \arrow[r, "\pi_1(j_1)"] \arrow[dr, bend right, "\phi_1"'] & \pi_1(X, x_0) \arrow[d, dashed, "\exists ! \phi"] & \pi_1(V, x_0) \arrow[l, "\pi_1(j_2)"'] \arrow[dl, bend left, "\phi_2"] \\ & H & \end{tikzcd}$$

To show ϕ is unique is easy. We know ϕ on $\text{im}(\pi_1(j_1))$ and $\text{im}(\pi_1(j_2))$ since the triangles above commute. By the baby S-vK Thm, these images generate the whole group $\pi_1(X, x_0)$ so ϕ is uniquely determined.

The hard part is showing ϕ exists; we'll outline the steps.

1) We define a function ρ that assigns to each loop f in either U or V an element of H:

$$\rho(f) = \begin{cases} \phi_1([f]_U) & \text{if } \text{im } f \subseteq U \\ \phi_2([f]_V) & \text{if } \text{im } f \subseteq V \end{cases}$$

where $[f]_U$ means "path homotopy class of f in U."
$[f]$ with no subscript means "path homotopy class of f in X."

To check that ρ is well-defined, suppose $\text{im}(f) \subseteq U \cap V$:

$$\begin{aligned}
\phi_1([f]_U) &= \phi(\pi_1(j_1)[f]_U) && \text{since the left triangle commutes} \\
&= \phi \circ \pi_1(j_1) \circ \pi_1(i_1)[f]_{U\cap V} && \text{since the top diamond commutes} \\
&= \phi \circ \pi_1(j_2) \circ \pi_1(i_2)[f] \\
&= \phi \circ \pi_1(j_2)[f]_V \\
&= \phi_2([f]_V) && \text{since the right triangle commutes.}
\end{aligned}$$

Note ρ satisfies

a) $[f]_U = [g]_U$ or $[f]_V = [g]_V \Rightarrow \rho(f) = \rho(g)$

b) if $\operatorname{im} f, \operatorname{im} g \subseteq U$ or $\operatorname{im} f, \operatorname{im} g \subseteq V$, then $\rho(f * g) = \rho(f)\rho(g)$

2) We extend ρ to a function σ that assigns to each path f in either U or V an element of H.

Check that a) + b) still hold where now f, g are paths.

3) Extend σ to a function τ that assigns to every path $f \in X$ an element of H.

Check

a') If $[f] = [g]$ for any paths f, g in X, then $\tau(f) = \tau(g)$

b') for all paths f, g in X, $\tau(f * g) = \tau(f)\tau(g)$ if the composite $f * g$ is defined.

This is the really hard part. From this point on, the proof is easy:

If f is a based loop in X, let $\phi([f]) = \tau(f)$.

Condition a′) implies that ϕ is well-defined; b′) implies that ϕ is a homomorphism. We just need to check that ϕ makes the triangles commute.

If f is a loop in U, then

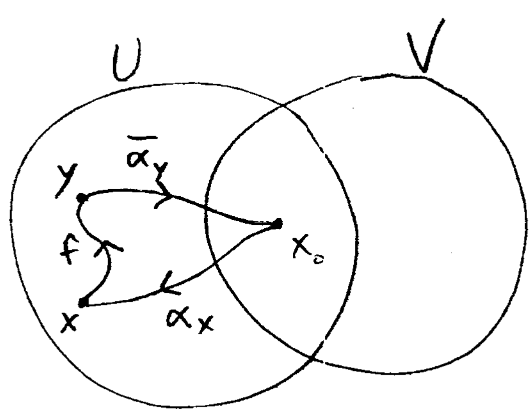
$$\phi \circ \pi_1(j_1)[f]_U = \phi[f] = \tau(f) = \rho(f)$$

since τ extends ρ and f is a loop in U. But

$$\rho(f) = \phi_1 [f]_U$$

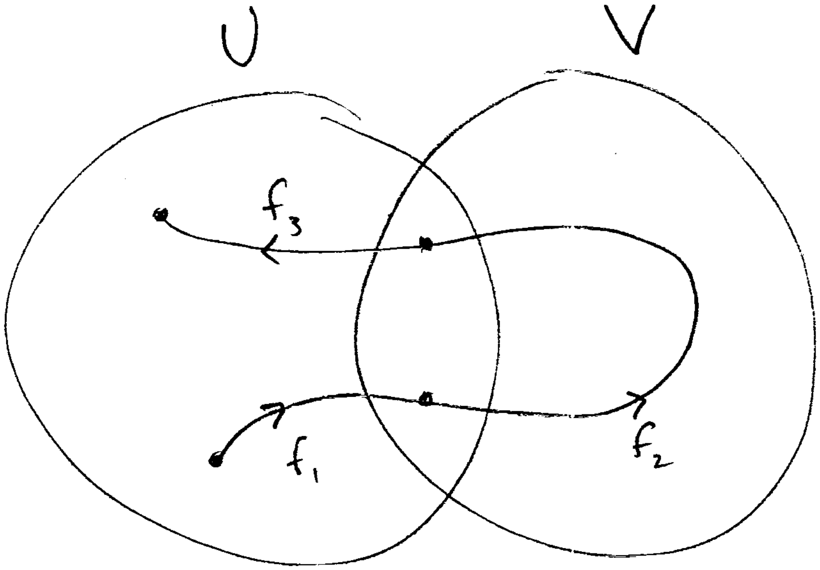
So the left-hand triangle commutes, and similarly for the right-hand triangle.

How we extend ρ (defined on <u>loops</u> either in U or in V) to σ (defined on <u>paths</u> either in U or in V):



For every point $x \in U$ we pick a path $\alpha_x : x_0 \to x$, and similarly for V. We can do this, because we assume U, V are path-connected. Then define $\sigma(f) = \rho(\alpha_x * f * \bar{\alpha}_y)$ if f is a path in U or V from x to y.

Now we extend σ to $\tilde{\tau}$ (defined on all paths in X):



Any path f in X can be written as $f_1 * \cdots * f_n$ where each f_i lies entirely in U or V, and then define

$$\tilde{\tau}(f) = \sigma(f_1) \cdots \sigma(f_n)$$

We need to check this is well-defined.