28 Feb 07

From Munkres' S-vK Thm to the Full-Fledged One

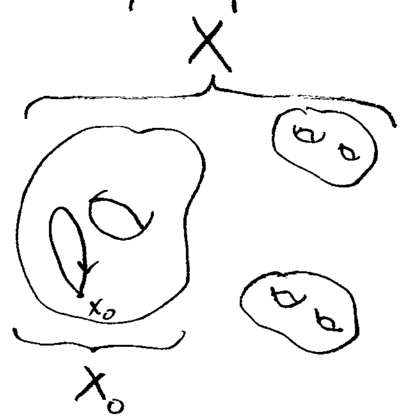
Let's assume (X, x_0) a pointed space, U & V open in X, $U \cap V$ is path connected and contains x_0, $U \cup V = X$. This is enough to imply the S-vk theorem, but Munkres also assumes U, V are path connected. How to go from his S-vK Thm to the full-fledged one?

Consider $X_0 := \{x \in X : \exists \text{ a path in } X \text{ from } x_0 \text{ to } x\}$ i.e. the <u>path component</u> of x_0. The inclusion

$X_0 \hookrightarrow X$ gives a homomorphism

$$\pi_1(X_0, x_0) \longrightarrow \pi_1(X, x_0)$$

which is actually an isomorphism, since any loop based at x_0 is contained in X_0. Note that X_0 is path-connected.



Similarly, we can form

$U_0 = U \cap X_0$
$V_0 = V \cap X_0$

And then

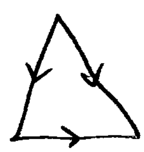
$$(U \cap V) \cap X_0 = U_0 \cap V_0 \quad \text{and} \quad U_0 \cup V_0 = (U \cap X_0) \cup (V \cap X_0) = (U \cup V) \cap X_0$$
$$= X \cap X_0 = X_0$$

Note that the inclusions $U_0 \hookrightarrow U$, $V_0 \hookrightarrow V$, $U_0 \cap V_0 \hookrightarrow U \cap V$ all give rise to isomorphisms of fundamental groups. So to prove the full-fledged S-vK theorem, it suffices to prove the special case considered by Munkres.

$$\begin{tikzcd} & U_0 \cap V_0 \arrow[dl, hook] \arrow[dr, hook] & \\ U_0 \arrow[dr, hook] & & V_0 \arrow[dl, hook] \\ & X_0 & \end{tikzcd}$$

Homework: show all these are path-connected. (Hint: draw some pictures of examples first.)

Last week I told you to compute π_1 of the <u>dunce cap</u>:

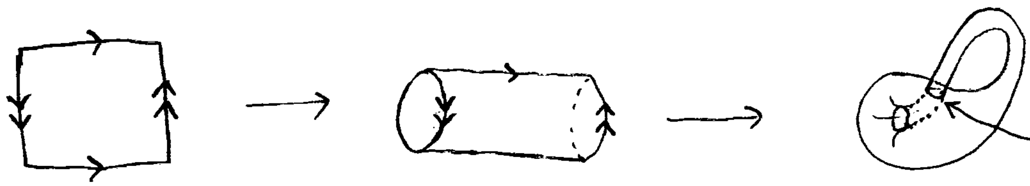


formed by identifying all three edges of a solid triangle as shown. What's it like?

very hard to draw, but embeddable in $\mathbb{R}^3$

(Warning! This is nothing like Munkres' dunce cap!)

The Klein bottle, on the other hand, can't be embedded in $\mathbb{R}^3$. This is obtained by identifying edges of a square like this:

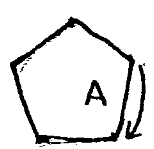


K

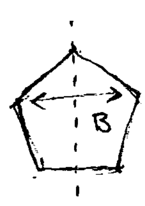
doesn't self-intersect: goes around this boundary into 4th dimension just like Möbius strip goes into third

$$\boxed{\text{Homework: compute fundamental group of } K.}$$

For each $n \geq 3$ there's a group D_n called the <u>nth dihedral group</u>: all symmetries of a regular n-gon, including rotations and reflections.



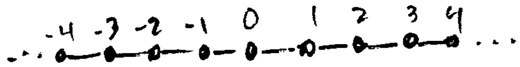
rotate clockwise by $\frac{2\pi}{n}$ $\qquad A^n = 1$



reflect $\qquad B^2 = 1$

$$BAB^{-1} = A^{-1}$$

There's also a group D_∞, "$\lim_{n\to\infty} D_n$" in some intuitive sense.
It's the symmetry group of



i.e. $\mathbb{Z}^1$

$$n \xmapsto{A} n+1$$

$$n \xmapsto{B} -n$$

So D_∞ is the group of bijections generated by A, B, i.e.

$$f(n) = \pm n + k \qquad k \in \mathbb{Z}$$

$\boxed{\text{Homework: find a homomorphism from } \pi_1(K,*) \text{ to } D_\infty.}$