ADVANCED TOPICS

We've talked about pushouts – these are examples of "colimits." There's also an important class of constructions called "limits." Limits & colimits are concepts that make sense in any category. Say we have some category; we can ask if there are objects with some particular universal property.

COLIMITS

pushout

(note that you can get from anywhere in this part both to $A +_C B$ and to Q and there's a unique morphism from $A +_C B$ to Q making the diagram commute)

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[dr] & \\
A \arrow[dr] \arrow[ddr, bend right] & & B \arrow[dl] \arrow[ddl, bend left] \\
 & A +_C B \arrow[d, dashed, "\exists!"] & \\
 & Q &
\end{tikzcd}$$

coproduct

ditto for $A+B, Q$

$$\begin{tikzcd}
A \arrow[dr] \arrow[ddr, bend right] & & B \arrow[dl] \arrow[ddl, bend left] \\
 & A + B \arrow[d, dashed, "\exists!"] & \\
 & Q &
\end{tikzcd}$$

more general colimit

ditto for X, Q

$$\begin{tikzcd}
 & B \arrow[dl] \arrow[dr] & & \\
A \arrow[dr] & & D \arrow[dl] \arrow[r] & E \arrow[dl] \arrow[ddl, bend left] \\
 & C \arrow[dr] \arrow[ddr, bend right] & & \\
 & & X \arrow[d, dashed, "\exists!"] & \\
 & & Q &
\end{tikzcd}$$

LIMITS

pullback

$$\begin{tikzcd}
 & Q \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & A \times_C B \arrow[dl] \arrow[dr] & \\
A \arrow[dr] & & B \arrow[dl] \\
 & C &
\end{tikzcd}$$

note that you can get to anywhere in this part both from $A \times_C B$ and from Q and there's a unique morphism $Q \to A \times_C B$ make the diagram commu

product

$$\begin{tikzcd}
 & Q \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & A \times B \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

ditto for $A \times B, Q$

more general limit

$$\begin{tikzcd}
 & Q \arrow[d, dashed, "\exists!"] \arrow[dd, bend right] \arrow[ddrr, bend left] & & \\
 & X \arrow[d] \arrow[drr] & & \\
 & C \arrow[dl] \arrow[dr] & & E \arrow[dl] \\
A \arrow[dr] & & D \arrow[dl] & \\
 & B & &
\end{tikzcd}$$

ditto for X, Q

	Set	Top	Top$_*$	Grp	AbGrp
product $A \times B$	$A \times B$ (cartesian product)	$A \times B$	$(A,a_0) \times (B,b_0) := (A \times B, (a_0,b_0))$	$A \times B = A \oplus B$	$A \oplus B$
coproduct $A + B$	$A + B = A \sqcup B$	$A + B = A \sqcup B$	$(A,a_0) \vee (B,b_0) = \frac{A \sqcup B}{a_0 \sim b_0}$	$A * B$ "free product"	$A \oplus B$

Set: sets & functions
Top : spaces & maps
Top$_*$: pointed spaces & pointed maps
Grp : groups & group homomorphisms
AbGrp: abelian groups & group homomorphisms

Given sets A, B, I claim their (categorical) product is the cartesian product:

$$\begin{tikzcd} & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ A & & B \end{tikzcd} \qquad \begin{aligned} p_1(a,b) &= a \\ p_2(a,b) &= b \end{aligned}$$

Given any set Q with functions $f_1 : Q \to A$, $f_2 : Q \to B$, there exists a unique function f to the product such that this diagram commutes:

$$\begin{tikzcd} & Q \arrow[dl, bend right, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, bend left, "f_2"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

$$\begin{aligned} p_1(f(q)) &= f_1(q) \\ p_2(f(q)) &= f_2(q) \end{aligned} \quad \Leftrightarrow \quad f(q) = (f_1(q), f_2(q))$$

In Set, $A+B$ is the disjoint union of A, B:

"A function $A \sqcup B \to Q$ is equivalent to a function $A \to Q$ and a function $B \to Q$"