Products & Coproducts of Pointed Spaces

Given two pointed spaces (A,a_0) & (B,b_0), what's their coproduct? A pointed space (X,x_0) with pointed maps

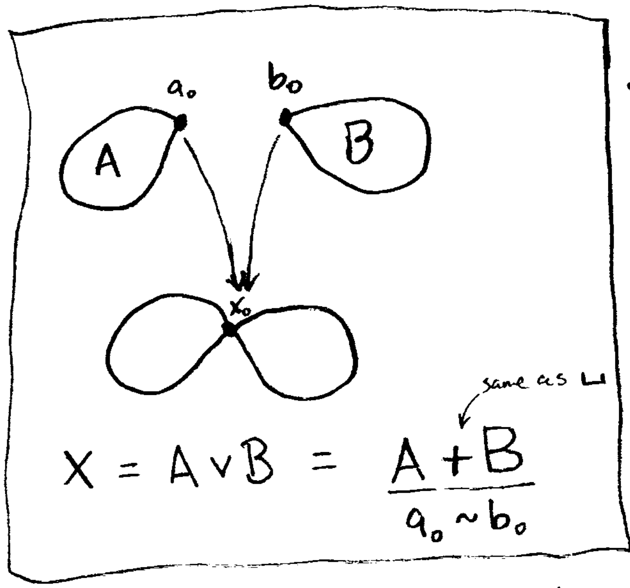
$$\begin{tikzcd} (A,a_0) \arrow[dr, "i_1"'] & & (B,b_0) \arrow[dl, "i_2"] \\ & (X,x_0) & \end{tikzcd}$$

with this universal property: for any other diagram

$$\begin{tikzcd} (A,a_0) \arrow[dr, "f_1"'] & & (B,b_0) \arrow[dl, "f_2"] \\ & (Q,q_0) & \end{tikzcd}$$

there exists a unique pointed map $f:(X,x_0)\to(Q,q_0)$ such that this diagram commutes:

$$\begin{tikzcd} (A,a_0) \arrow[r, "i_1"] \arrow[dr, "f_1"'] & (X,x_0) \arrow[d, "f"] & (B,b_0) \arrow[l, "i_2"'] \arrow[dl, "f_2"] \\ & (Q,q_0) & \end{tikzcd}$$



We claim that we can take X to be the quotient space of the disjoint union $A+B$ by the equivalence relation $a_0 \sim b_0$. Then X has a point $x_0 = [a_0] = [b_0]$; we claim (X, x_0) is the coproduct of (A, a_0) & (B, b_0). We define

$$i_1(a) = [a] \in X \quad \forall a \in A$$
$$i_2(b) = [b] \in X \quad \forall b \in B$$

Now we need to check the universal property. Given f_1, f_2 as above, let

$$f : (X, x_0) \to (Q, q_0)$$
$$i_1(a) \longmapsto f_1(a) \quad \forall a \in A$$
$$i_2(b) \longmapsto f_2(b) \quad \forall b \in B$$

This uniquely determines f; we get existence by showing it's 1) well-defined 2) continuous 3) pointed. when $i_1(a) = i_2(b)$. This occurs only when $a = a_0$ and $b = b_0$, and since $f_1(a_0) = f_2(b_0) = q_0$ it's well-defined.
The function f is continuous: we can use the pasting lemma after checking that $\operatorname{im} i_1$, $\operatorname{im} i_2 \subseteq X$ are closed & we defined f to be continuous on each of these. The function f is pointed: $f(x_0) = f(i_1(a_0)) = f_1(a_0) = f(i_2(b_0)) = f_2(b_0) = q_0$. So f exists.

The equations

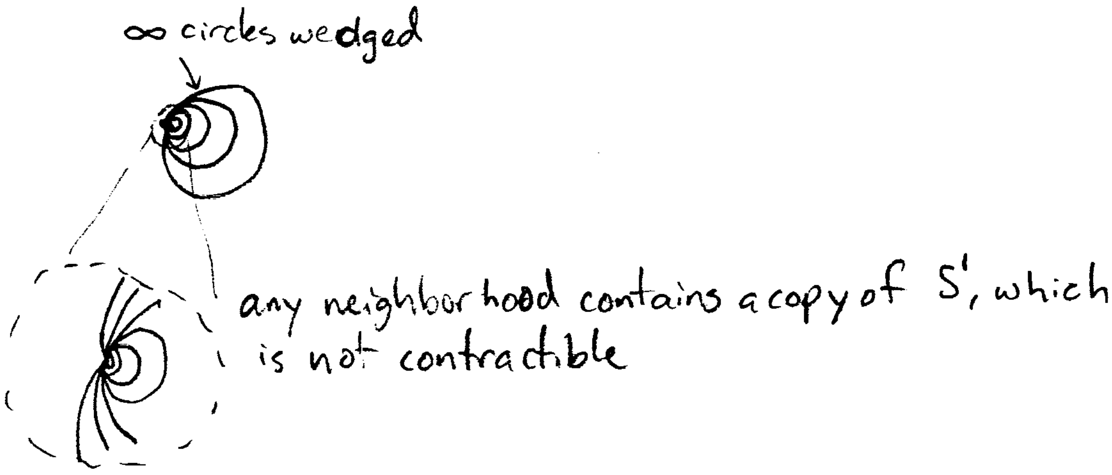
$$f(i_1(a)) = f_1(a) \qquad \forall a \in A$$
$$f(i_2(b)) = f_2(b) \qquad \forall b \in B$$

say the big diagram commutes, so we've seen $\exists!$ pointed map f making the diagram commute.

<u>Defn</u>: The <u>wedge</u> of pointed spaces $(A, a_0), (B, b_0)$ is their coproduct; it's denoted $(A, a_0) \vee (B, b_0)$ [or often $A \vee B$].

<u>Defn</u>: A space (A, a_0) is <u>well-pointed</u> if a_0 has an open neighborhood that's contractible.

Ex. The "Hawaiian earring" is not well-pointed:



<u>Thm</u>: Suppose $(A, a_0), (B, b_0)$ are well-pointed. Then

$$\pi_1\big((A, a_0) \underbrace{\vee}_{\text{coproduct of ptd spaces}} (B, b_0)\big) = \pi_1(A, a_0) \underbrace{*}_{\text{coproduct of groups}} \pi_1(B, b_0)$$

or, better,

$$\pi_1\big((A, a_0) + (B, b_0)\big) = \pi_1(A, a_0) + \pi_1(B, b_0).$$

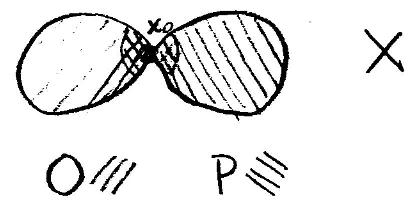
"π_1 preserves coproducts of well-pointed spaces."

Pf sketch: Let U be a contractible neighborhood of a_0, $U \subseteq A$,
V 〃 〃 b_0, $V \subseteq B$.

Consider

$$\begin{tikzcd}
(A,a_0) \arrow[dr, "i_1"'] & & (B,b_0) \arrow[dl, "i_2"] \\
& (A,a_0) \vee (B,b_0) = (X,x_0) &
\end{tikzcd}$$

Let
$$O = i_1(A) \cup i_2(V)$$
$$P = i_2(B) \cup i_1(U)$$



Check hypotheses of S-vK

1. O, P open
2. $O \cup P = X$
3. $O \cap P$ contains x_0
4. $O \cap P$ path-connected - true since $i_1(U)$, $i_2(V)$ contractible and $O \cap P = i_1(U) \cup i_2(V)$

So we have a pushout

$$\begin{tikzcd}
& \pi_1(O \cap P, x_0) = 1 \text{ trivial gp since } O\cap P \text{ contractible} \arrow[dl] \arrow[dr] & \\
\underset{\text{by deformation retract}}{\pi_1(A,x_0)} = \pi_1(O,x_0) \arrow[dr] & & \pi_1(P,x_0) = \pi_1(B,x_0) \text{ by deformation retract} \arrow[dl] \\
& \pi_1(X,x_0) &
\end{tikzcd}$$

So $\pi_1(X,x_0) = \pi_1(A,a_0) * \pi_1(B,b_0)/N$
where in this case N is trivial.