<u>Theorem</u> – The product of pointed spaces (A, a_0) & (B, b_0) is $(A\times B, (a_0, b_0))$.

Proof -

We have pointed maps

$$\begin{tikzcd} & (A\times B, (a_0, b_0)) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ (A, a_0) & & (B, b_0) \end{tikzcd}$$

$p_1(a, b) = a$
$p_2(a, b) = b$

We need to show that given any other diagram

$$\begin{tikzcd} & (Q, q_0) \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ (A, a_0) & & (B, b_0) \end{tikzcd}$$

there exists a unique pointed map $f: (Q, q_0) \to (A\times B, (a_0, b_0))$ such that this diagram commutes:

$$\begin{tikzcd} & (Q, q_0) \arrow[ddl, bend right, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[ddr, bend left, "f_2"] & \\ & (A\times B, (a_0, b_0)) \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ (A, a_0) & & (B, b_0) \end{tikzcd}$$

The fact that the triangles commute forces f to be

$$f(q) = (f_1(q), f_2(q))$$

so f is unique, but f also exists: it makes the diagram commute and it's a continuous pointed map: $f(q_0) = (a_0, b_0)$.

$\square$

We've seen that $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ preserves <u>some</u> pushouts (SvK thm) and <u>some</u> coproducts (namely coproducts of well-pointed spaces). But it preserves <u>all</u> products:

<u>Thm</u>: $\pi_1(A \times B, (a_0, b_0)) \cong \pi_1(A, a_0) \times \pi_1(B, b_0)$

Proof sketch: There's a 1-1 correspondence between loops γ in $(A \times B, (a_0, b_0))$ and pairs of loops (γ_1, γ_2) where γ_1 is a loop in (A, a_0) and γ_2 is a loop in (B, b_0). A loop γ in $(A \times B, (a_0, b_0))$ is a map

$$\gamma : (S^1, *) \to (A \times B, (a_0, b_0))$$

and such maps are in 1-1 correspondence with pairs (γ_1, γ_2) of ptd. map where

$$\gamma_1 : (S^1, *) \to (A, a_0)$$

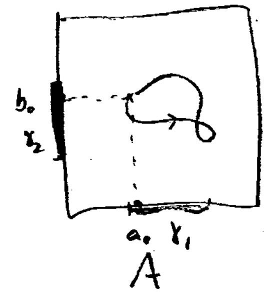
$$\gamma_2 : (S^1, *) \to (B, b_0)$$

i.e.

$$\gamma_i = p_i \circ \gamma$$

Given (γ_1, γ_2), we get γ via the universal property of the product

$$\begin{tikzcd}
 & (S^1, *) \arrow[d, dashed, "\exists! \gamma"] \arrow[ddl, bend right, "\gamma_1"'] \arrow[ddr, bend left, "\gamma_2"] & \\
 & (A \times B, (a_0, b_0)) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
(A, a_0) & & (B, b_0)
\end{tikzcd}$$



We'll write " $\gamma = (\gamma_1, \gamma_2)$ " if γ corresponds to the pair (γ_1, γ_2) in the way described. Check that

$$(\gamma_1, \gamma_2) * (\gamma_1', \gamma_2') = (\gamma_1 * \gamma_1', \gamma_2 * \gamma_2')$$

Also, there's a 1-1 correspondence between path homotopies in $(A \times B, (a_0, b_0))$

$$F : (S^1 \times I, *) \to (A \times B, (a_0, b_0))$$

and pairs (F_1, F_2) of path homotopies between loops in A, B respectively

$$F_1 : (S^1 \times I, *) \to (A, a_0)$$
$$F_2 : (S^1 \times I, *) \to (B, b_0).$$

Again $F_i = p_i \circ F$ and you get F from F_1, F_2 by the universal property

So: path homotopy classes of loops in $A \times B$ are in 1-1 correspondence with pairs consisting of a path homotopy class of loops in A and
" " " " " " " B. I.e.
we have a bijection of sets

$$\pi_1(A \times B, (a_0, b_0)) \cong \pi_1(A, *) \times \pi_1(B, *).$$

The equation

$$(\gamma_1, \gamma_2) * (\gamma_1', \gamma_2') = (\gamma_1 * \gamma_1', \gamma_2 * \gamma_2')$$

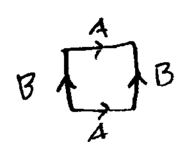
implies that the bijection is also a group isomorphism.

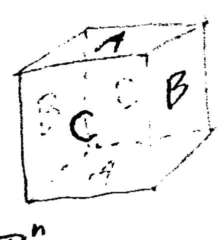
$\square$

This proof was so much easier because loops are maps <u>into</u> our space and the universal property of the product specifies a unique map into the product.

Example: The n-torus $T^n = S^1 \times \dots \times S^1$

$T^1 =$  S^1

$T^2 =$  

$T^3 =$ doesn't embed in $\mathbb{R}^3$ 

$$\pi_1(T^n, *) \cong \pi_1((S^1)^n, *) \cong \mathbb{Z}^n$$

Undoing π_1

We've been studying this functor

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}.$$

Is there a way of going back from Grp to Top_*? I.e. given a group G, can we find a pointed space whose fundamental group is isomorphic to G? Yes! And because there's a systematic way, we even get a functor

$$K(-,1) : \mathrm{Grp} \longrightarrow \mathrm{Top}_*$$

such that

$$\pi_1(K(G,1),*) \cong G.$$

This is called the Eilenberg-Maclane space.