9 Mar '07

HW 73.1 <u>Don't</u> use Eilenberg-MacLane spaces; use methods you already know!

74.2

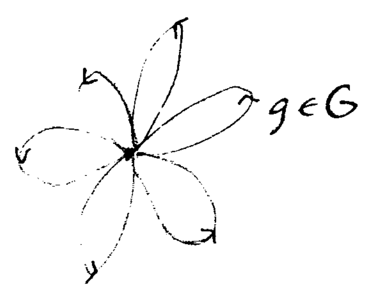
Eilenberg-MacLane Spaces

Given a group G, Eilenberg & MacLane (inventors of categories in 1947) invented a pointed space $K(G,1)$ with $\pi_1(K(G,1),*) \cong G$. Here's how we build $K(G,1)$, in stages:

0) Let $X_0 = \{*\}$

   $\pi_1(X_0,*) \cong 1$

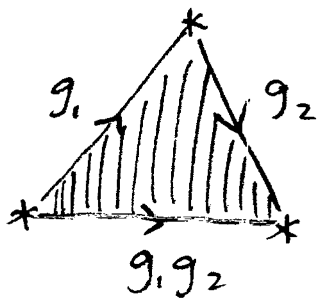
1) Build X_1 by attaching an "edge" to X_0 for each $g \in G$:



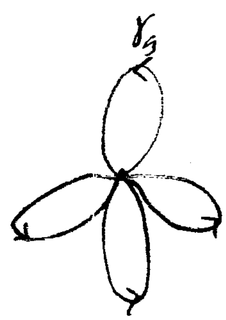
X_1 is a bouquet of $|G|$ circles. $\pi_1(X,*) \cong F_{|G|}$, the free group on $|G|$ generators. (This can be defined even if $|G|$ is infinite - see Munkres for details.)

$F_{|G|} \not\cong G$ since G also has relations, not just generators, so

2) Build X_2 by attaching a triangle to X_1 for each $g_1, g_2 \in G$:



In X_1, we have a loop γ_g for each $g \in G$:



but in X_1 $\gamma_{g_1} * \gamma_{g_2} \not\simeq \gamma_{g_1 g_2}$. However, in X_2, we filled in a triangle between them, so in $X_2 \supset X_1$, we do have $\gamma_{g_1} * \gamma_{g_2} \simeq \gamma_{g_1 g_2}$.

Taking $\pi_1(X_2, *)$, we therefore have

$$[\gamma_{g_1}] * [\gamma_{g_2}] = [\gamma_{g_1 g_2}].$$

In fact, $\pi_1(X_2, *)$ will have $[\gamma_g]$ as generators and (only) these relations, i.e.

$$\pi_1(X_2,*) \cong F_{|G|} \big/ \langle [\gamma_{g_1}] * [\gamma_{g_2}] * [\gamma_{g_1 g_2}]^{-1} \rangle$$

or in another standard notation,

$$\pi_1(X_2,*) \cong \langle \underbrace{[\gamma_g] \ \forall g \in G}_{\text{generators}} \ | \ \underbrace{[\gamma_{g_1}][\gamma_{g_2}] = [\gamma_{g_1 g_2}]}_{\text{relations}} \ \forall g_1 g_2 \in G \rangle$$

or, since G has the presentation

$$G = \langle g \ \ \forall g \in G \ | \ g_1 \cdot g_2 = g_1 g_2 \quad \forall g_1 g_2 \in G \rangle$$

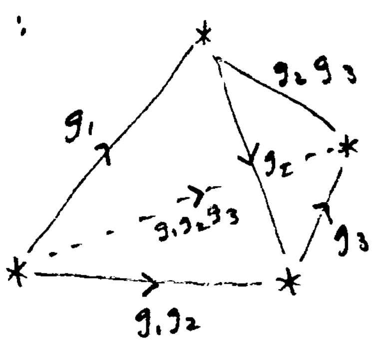
and there's an obvious isomorphism between them,

$$\pi_1(X_2,*) \cong G \, .$$

We're done if we just wanted a space whose π_1 is G. But X_2 has a very complicated π_2. Recall

$$\pi_n(X,*) = \{ \text{homotopy classes of pointed maps } f:(S^n,*) \to (X,*) \}.$$

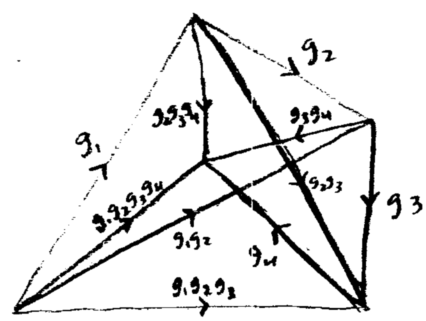
Loosely, π_n counts "n-dimensional holes" in X, and X_2 has lots of 2-dimensional holes, one inside of each tetrahedron:



3) We add a tetrahedron for each triple $g_1 g_2 g_3$ to build X_3. Now $\pi_2(X_3, *) \cong 1$, but $\pi_3(X_3, *)$ is complicated.

⋮

n) Build X_n by gluing in an n-simplex for each n-tuple. Now $\pi_{n-1}(X_n, *) \cong 1$ but $\pi_n(X_n, *)$ is complicated.



$n=4$

⋮

We get spaces $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$

so let

$$\bigcup_{n=0}^{\infty} X_n = K(G,1),$$

the <u>Eilenberg-MacLane Space</u> of G. "1" means only π_1 is nontrivial.

<u>Thm</u> For any group G,

$$\pi_1(K(G,1),*) \cong G$$

and

$$\pi_n(K(G,1),*) \cong 1 \quad \forall n>1.$$

" $K(G,1)$ is the space that best personifies G "

i.e. it has no extra structure.