What's more fundamental than the fundamental group?

To avoid needing a basepoint, we should use the fundamental groupoid:

Defn: A groupoid is a category where every morphism

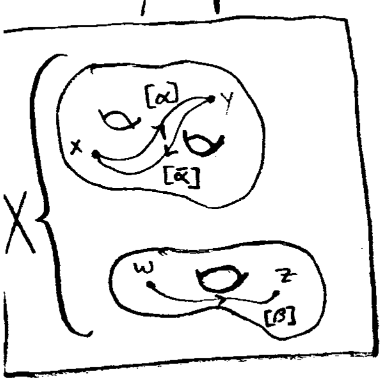
$$f: x \to y$$

has an inverse, ie. some $g: y \to x$ such that

$$g \circ f = 1_x$$
$$f \circ g = 1_y$$

(If a morphism f has an inverse, it's unique, so we can call it $f^{-1}: y \to x$.)

Any space X gives a groupoid where



- objects are points
- morphisms are path homotopy classes of paths
- the inverse of any morphism $[\alpha]: x \to y$ is the reverse $[\bar{\alpha}]: y \to x$ since $\alpha * \bar{\alpha}$ is contractible.

called the fundamental groupoid $\Pi_1(X)$. It has
capital π

- composition $g \circ f = f * g$ concatenation of paths
- identities $1_x: x \to x$ constant path staying at x
- $f: x \to y = f \circ 1_x = 1_y \circ f$

Prop: Given a groupoid G and any object $x \in G$, there's a group $\mathrm{End}(x)$ of morphisms $f: x \to x$ where multiplication is composition of morphisms. Conversely, given a group G, there's a groupoid with one object $*$ and morphisms $f: * \to *$ for each $f \in G$, where composition is multiplication of elements.

Pf. Follows from the definitions.

$\square$

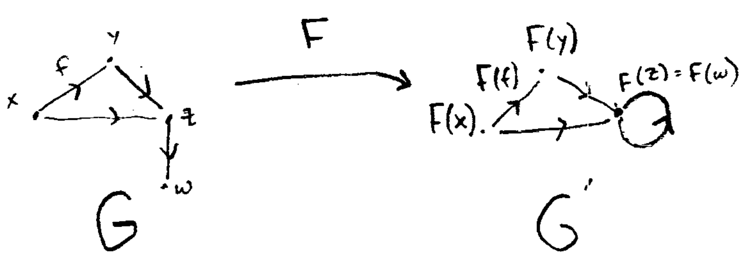
Moral: "A group is a groupoid with one object."

$$\text{Group Theory} \subseteq \text{Groupoid Theory}$$

$\Pi_1(X)$ is better than $\pi_1(X,*)$ because it doesn't need a basepoint and because it contains $\pi_1(X,x_0)$ $\forall x_0 \in X$. This simplifies life, especially the S-vK thm.

Defn - The category Gpd has

- objects groupoids
- morphisms functors



a morphism $F: G \to G'$ in Gpd

For any space X, $\Pi_1(X)$ is a groupoid, but also

For any map $m: X \to X'$ we get a functor $\Pi_1(m): \Pi_1(X) \to \Pi_1(X')$

The picture explains how

- given an object $x \in \Pi_1(X)$ i.e. a point $x \in X$,
  we define $\Pi_1(m)(x) = m(x)$

- given a morphism $g = [\gamma] \in \Pi_1(X)$, we
  define $\Pi_1(m)(g) = [m \circ \gamma]$

<u>Prop</u> $\Pi_1 : \mathrm{Top} \to \mathrm{Gpd}$ is a functor

Pf sketch.

Check

$$\Pi_1(g \circ f) = \Pi_1(g) \circ \Pi_1(f)$$

$$\Pi_1(1_X) = 1_{\Pi_1(X)}$$

$\square$

<u>Better SvK Theorem.</u> If X is a space, $U, V \subseteq X$ open,

$U \cup V = X$, then

$$\begin{tikzcd}
 & U \cap V \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\
U \arrow[dr, "j_1"'] & & V \arrow[dl, "j_2"] \\
 & \begin{matrix} U \cup V \\ = X \end{matrix} &
\end{tikzcd}$$

is a pushout in Top, and

$$\begin{tikzcd}
& \pi_1(U \cap V) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\pi_1(U) \arrow[dr, "\pi_1(j_1)"'] & & \pi_1(V) \arrow[dl, "\pi_1(j_2)"] \\
& \pi_1(U \cup V) &
\end{tikzcd}$$

is a pushout.

Pf. Left to reader.

$\square$

This is much cleaner! There's no mention of basepoints or connectedness.