14 Mar '07

Classifying Space of a Groupoid.

We've seen that for any group G, there's a space $K(G,1)$, the Eilenberg-MacLane space, with

$$\pi_1(K(G,1),*) \cong G \quad \text{and} \quad \pi_n(K(G,1),*) = 0 \quad \forall n>1.$$

But groups are just groupoids with one object, and π_1 is a baby version of

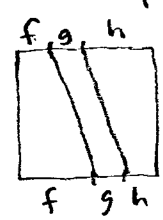
$$\Pi_1 : \mathrm{Top} \to \mathrm{Gpd}$$

(I'm going to leave off the serifs now...)

which to any space X gives a groupoid $\Pi_1(X)$ with

- points of X as objects
- homotopy classes of paths in X as morphisms

because composition isn't associative

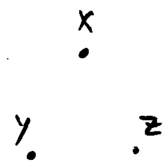


associator homotopy

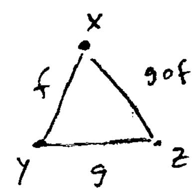
In fact, we can generalize this construction to turn groupoids into spaces. Given a groupoid G, we get a space that *should* be called $K(G,1)$ but is called the <u>classifying space</u> of G, BG. How do we build BG?

Just like the Eilenberg-MacLane space, but:

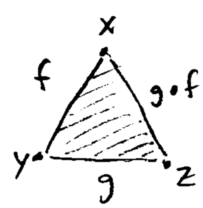
0) We first build X_0, a discrete space with one point for each object of G:



1) Next we build X_1 by adding one edge for each morphism: given $f: x \to y$, we add an edge $x — y$ (undirected, but that's OK because we're in a groupoid)

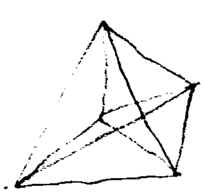


2) We build X_2 by adding a solid triangle for each composable pair of morphisms



⋮

n) Add a solid n-simplex for each composable n-tuple of morphisms to get X_n



⋮

We get inclusions

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

so let $BG = \bigcup_{i=0}^{\infty} X_i$ with topology st. U open in BG iff $U \cap X_i$ is open in X_i.

Example: Suppose $G = FinSet_0$, the groupoid where

- objects are finite sets
- morphisms are bijections

Then BG is an important space related in a cool way to spheres (but it's too complicated to explain here), and also

$$BG \simeq \coprod_{n=0}^{\infty} K(S_n, 1)$$

where S_n is the group of permutations of n elements.

This makes sense, since G is a disjoint union of "connected components," each component being made up of all n-element sets connected by permutations.