W 10 Jan
Derek

We defined the fundamental group functor

$$\pi_1 : \text{Pointed Spaces} \longrightarrow \text{Groups}$$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

(pointed space) (group of homotopy classes of loops in X with mult $[f]*[g] = [f*g]$)

$$h : (X, x_0) \to (Y, y_0) \longmapsto \pi_1(h) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

$$[f] \longmapsto [h \circ f]$$

(map of pointed spaces i.e. $h(x_0) = y_0$) (homomorphism of groups)

This is a functor because

$$\pi_1(h \circ k) = \pi_1(h) \circ \pi_1(k)$$

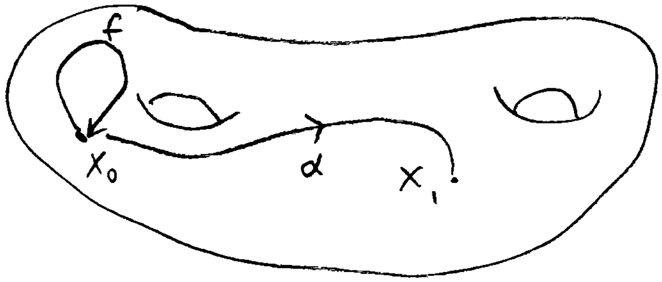
$$\pi_1(1_{(X, x_0)}) = 1_{\pi_1(X, x_0)}$$

It was stated but not proven that $\pi_1(h)$ is a homomorphism.

Pf:
$$\begin{aligned}
\pi_1(h)([f]*[g]) &= \pi_1(h)([f*g]) \\
&= [h \circ (f*g)] \\
&= [(h \circ f)*(h \circ g)] \quad \text{check this step!} \\
&= [h \circ f]*[h \circ g] \\
&= \pi_1(h)(f) * \pi_1(h)(g)
\end{aligned}$$

Next: How are $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ related?

Suppose α is a path with $\alpha(0) = x_0$ $\alpha(1) = x_1$.



Define a function

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$

Thm 52.1 $\quad \hat{\alpha}$ is a group isomorphism.

Pf.
$$\begin{aligned}
\hat{\alpha}([f] * [g]) &= [\bar{\alpha} * f * g * \alpha] \\
&= [\bar{\alpha} * f] * [e_{x_0}] * [g * \alpha] \\
&= [\bar{\alpha} * f * \alpha * \bar{\alpha} * g * \alpha] \\
&= [\bar{\alpha} * f * \alpha] * [\bar{\alpha} * g * \alpha] \\
&= \hat{\alpha}([f]) * \hat{\alpha}([g])
\end{aligned}$$
so $\hat{\alpha}$ is a homomorphism.

$\hat{\alpha}$ has $\hat{\bar{\alpha}}$ as its inverse:

$$\hat{\alpha} \circ \hat{\bar{\alpha}}([f]) = [\bar{\alpha} * (\underset{\alpha}{\underset{\parallel}{\bar{\bar{\alpha}}}} * f * \bar{\alpha}) * \alpha]$$

$$= [f]$$

and similarly for $\hat{\bar{\alpha}} \circ \hat{\alpha}$.

$\square$

Defn: If X is a path-connected space, X is called __simply connected__ if $\pi_1(X, x_0) = 0$ for some (and hence every) $x_0 \in X$.

Examples:

$D^2 =$  is simply connected

$\mathbb{R}^n$ is simply connected

$\mathbb{R}^2 - \{0\}$ is __not__ simply connected: 

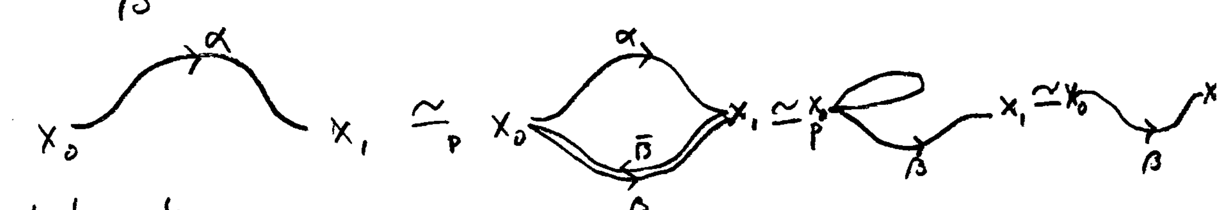
$\mathbb{R}^3 - \{0\}$ __is__ simply connected

$\mathbb{R}^3 -$ line is not s.c.

S^1 is not s.c

S^2 is s.c.

__Lem 52.4:__ Let X be s.c. Then any two paths α, β $x_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} x_1$ are path-homotopic, where $x_0, x_1 \in X$.

Pf. 

or, algebraically

$$[\alpha] = [\alpha] * [e_{x_1}] = [\alpha] * [\bar{\beta}] * [\beta] \underset{\text{by s.c.}}{=} [e_{x_0}] * [\beta] = [\beta]$$

$\square$

Next goal: $\pi_1(S^1, x_0) \cong \mathbb{Z}$.

For this, we need the idea of

COVERING SPACES

<u>Defn:</u> Let $p: E \to B$ be a surjective map of spaces. An open set $U \subset B$ is called <u>evenly covered</u> by p if

$$p^{-1}(U) = \bigcup_{\alpha} V_\alpha$$

where V_α are disjoint open sets in E, called <u>slices</u>, and

$$p\big|_{V_\alpha} : V_\alpha \to U$$

is a homeomorphism for each α.

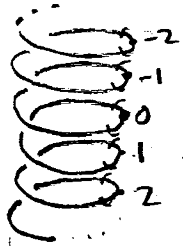
<u>Defn:</u> If every $b \in B$ has a neighborhood that is evenly covered by p, then p is called a <u>covering map</u>,
E is called a <u>covering space</u> of B, and
B is called the <u>base space</u>.

Ex (Thm 53.1) Identify S^1 with unit complex numbers. The map

$$p: \mathbb{R} \to S^1$$
$$x \mapsto e^{2\pi i x}$$

is a covering map.

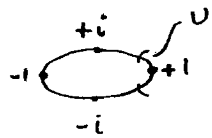
$E = \mathbb{R}$



— p restricted to any one of these intervals V_α is a homeomorphism

$\downarrow$

$B = S^1$



Other covering spaces:

$$\begin{array}{c} E = S^1 \\ \downarrow \\ B = S^1 \end{array}$$

$$\begin{array}{c} E = S^1 \\ \downarrow \\ B = S^1 \end{array}$$

$$\begin{array}{c} z \\ \big\downarrow \\ z^2 \end{array}$$

$$\begin{array}{c} E = S^1 \amalg S_1 \\ \downarrow \\ B = S^1 \end{array}$$

$\underbrace{\hspace{10em}}_{\text{two-fold covers}}$