17 Jan 07
John's back from Toronto

Goal: calculate fundamental groups, e.g.

$$\pi_1(S^1, *) = \mathbb{Z}$$

Tools:

1) <u>Defn:</u> A map $p: E \to B$ is a <u>covering (map)</u> if it's surjective & each $b \in B$ has a nbhd $U \ni x$ such that $p^{-1}(U)$ is a disjoint union of spaces $\{V_\alpha\}_{\alpha \in A}$ called <u>sheets</u> s.t. $p|_{V_\alpha}: V_\alpha \to U$ is a homeomorphism $\forall \alpha \in A$. $\underset{E}{\overset{V_\alpha}{\cap}}$

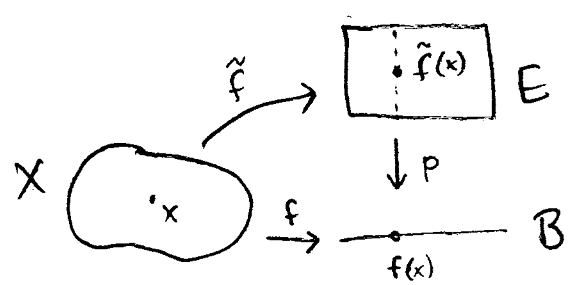
$E \downarrow B$ 

We say U is <u>evenly covered</u> and we call E a <u>covering space</u> of B. B is called the <u>base (space)</u> and p is the <u>projection</u>.

2) <u>Defn:</u> Given a map $p: E \to B$ and a map $f: X \to B$, we say $\tilde{f}: X \to E$ s.t. $p \circ \tilde{f} = f$ is a <u>lifting</u> of f, or that $\tilde{f}$ <u>lifts</u> f <u>along</u> p. I.e.

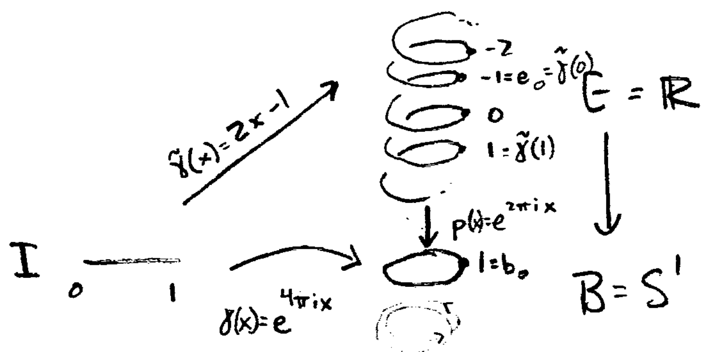
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.



Thm 54.3: Suppose $p: E \to B$ is a covering map, let $b \in B_0$, and suppose $e_0 \in E$ has $p(e_0) = b_0$. Suppose $\gamma: I \to B$ is a loop based at b_0. Then there exists a unique lift of γ along p to a path $\tilde{\gamma}: I \to E$ s.t. $\tilde{\gamma}(0) = e_0$. If γ_0, γ_1 are path homotopic then $\tilde{\gamma}_0, \tilde{\gamma}_1$ are path homotopic & thus $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

E.g.



Changing γ by a path homotopy doesn't change $\tilde{\gamma}(1)$,

So we get a map

$$\begin{aligned} \phi: \pi_1(B, b_0) &\longrightarrow p^{-1}(b_0) \subseteq E \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

which in the example above gives

$$\phi: \pi_1(B, b_0) \longrightarrow \mathbb{Z}$$

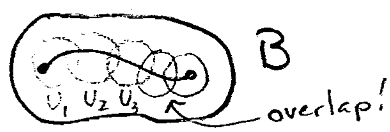
which will turn out to be an isomorphism!

Lem 54.1: Suppose $p: E \to B$ covering map, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, $\gamma: I \to B$ a path. $b_0 \xrightarrow{\gamma} b_1$

$\gamma(0) = b_0$. Then γ has a unique lift $\tilde{\gamma}: I \to E$ along p.

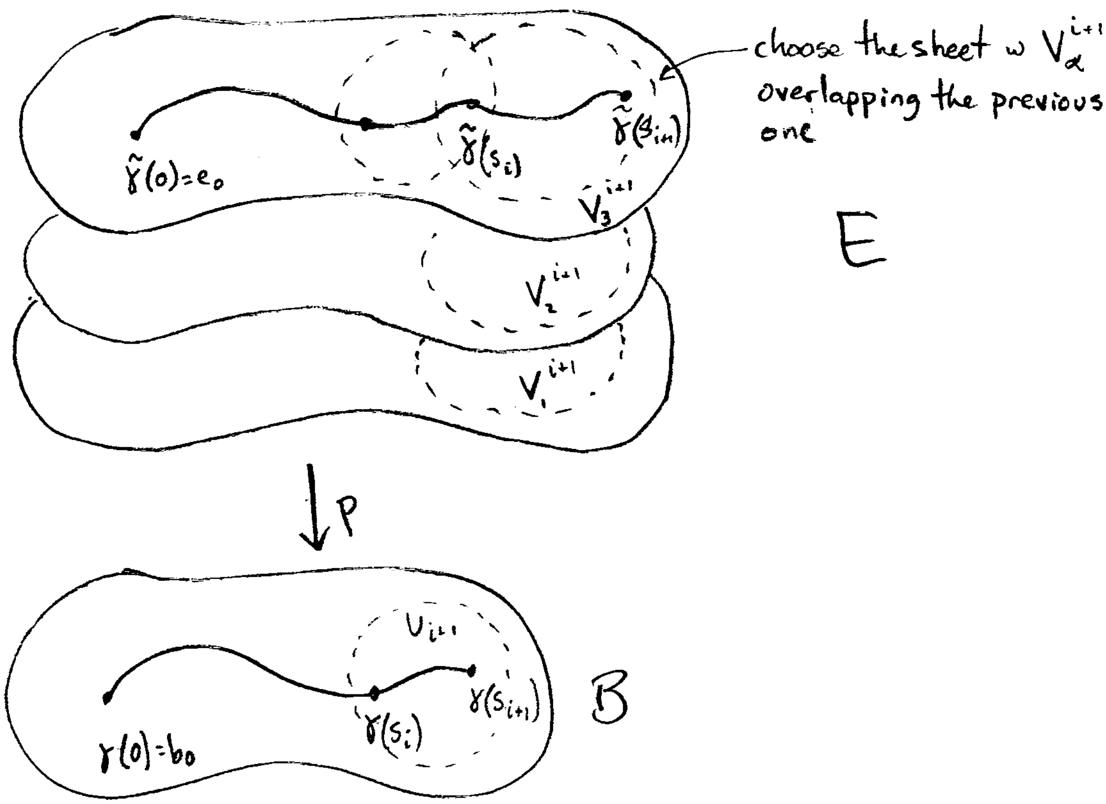
Pf. Existence: choose for each point $b \in B$ an evenly covered neighborhood to get an open cover of B. Since I is compact, $\gamma(I)$ is compact and thus there's a finite open subcover of $\gamma(I)$.

By the Lebesgue number theorem (whatever Munkres calls it), we can find $0 = s_0 \leq s_1 \leq \cdots \leq s_n = 1$ s.t. each interval $[s_{i-1}, s_i]$ $(1 \leq i \leq n)$ has $\gamma([s_{i-1}, s_i]) \subseteq U_i$ for one of the open sets in our cover.



Suppose we've succeeded in lifting $\gamma\big|_{[0,s_i]}: [0, s_i] \to B$ to $\tilde{\gamma}\big|_{[0,s_i]}: [0, s_i] \to E$.

(For $i = 0$ we have $\gamma(0) = b_0$ and define $\tilde{\gamma}(0) = e_0$.) For $i > 0$,



For $s \in [s_i, s_{i+1}]$ we must choose $\tilde{\gamma}(s)$ s.t. $p \circ \tilde{\gamma}(s) = \gamma(s)$, so $\tilde{\gamma}(s) \in p^{-1}(U) = \bigsqcup_\alpha V_\alpha$ with $p|_{V_\alpha}$ homeomorphism. For $\tilde{\gamma}$ to be continuous, $\hat{\gamma}(s)$ must lie in one of the V_α, namely the one that overlaps with the previous one.