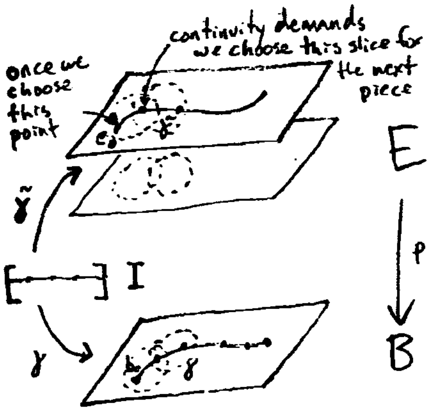
19 Jan 07



<u>Lemma 54.1</u>: Suppose $p: E \to B$ is a covering map & $p(e_0) = b_0$. Suppose $\gamma: [0,1] \to B$ is a path w/ $\gamma(0) = b_0$. Then γ has a unique lift along p to $\tilde{\gamma}: [0,1] \longrightarrow E$ with $\tilde{\gamma}(0) = e_0$.

Proof of uniqueness – again show inductively that there's a <u>unique</u> $\tilde{\gamma}\big|_{[s_i, s_{i+1}]}$ s.t. it lifts $\gamma\big|_{[s_i, s_{i+1}]}$ & $\tilde{\gamma}\big|_{[s_i, s_{i+1}]}(s_i) = \tilde{\gamma}\big|_{[s_{i-1}, s_i]}(s_i)$.
To do this, note that

$$\gamma[s_i, s_{i+1}] \subset U_i \qquad U_i \text{ evenly covered}$$
$$\tilde{\gamma}[s_i, s_{i+1}] \text{ must lie in } p^{-1}(U_i) = \bigsqcup V_\alpha$$

Since we need $\tilde{\gamma}$ to be continuous, $\tilde{\gamma}[s_i, s_{i+1}] \subset V_{\alpha_0}$ for <u>one</u> α_0, & this must be α_0 s.t. $\tilde{\gamma}\big|_{[s_{i-1}, s_i]}(s_i) \in V_{\alpha_0}$. Since $p|_{V_{\alpha_0}}: V_{\alpha_0} \longrightarrow U_i$ is a homeomorphism and we want

$$p|_{V_{\alpha_0}} \circ \tilde{\gamma}\big|_{[s_i, s_{i+1}]} = \gamma\big|_{[s_i, s_{i+1}]} \qquad \tilde{\gamma} \text{ lifts } \gamma$$

we must have

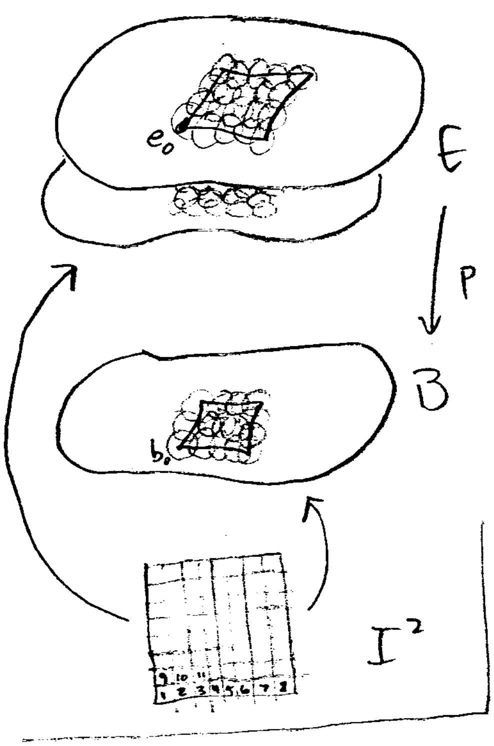
$$\tilde{\gamma}\big|_{[s_i, s_{i+1}]} = \left(p|_{V_{\alpha_0}}\right)^{-1} \circ \gamma\big|_{[s_i, s_{i+1}]}.$$

In short, $\tilde{\gamma}\big|_{[s_i, s_{i+1}]}$ is uniquely determined by induction.
So γ has a unique lift $\tilde{\gamma}$ s.t. $\tilde{\gamma}(0) = e_0$

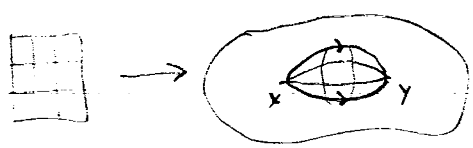
Hw due 1/26
54-7 see ex 4 p.339
55-4acd copy proofs for $n=1$

<u>Lemma 54.2</u>: Same assumptions - $p: E \to B$ covering map, $b_0 \in B$, $e_0 \in E$, $p(e_0) = b_0$. Suppose $F: [0,1]^2 \to B$ has $F(0,0) = b_0$. Then there exists a unique lift of F along p to $\tilde{F}: [0,1]^2 \to E$ s.t. $\tilde{F}(0,0) = e_0$.
If F is a path homotopy, so is $\tilde{F}$.

Proof sketch: First part very much like Lemma 54.1 Cover B with evenly covered neighborhoods (Lebesgue number lemma 21.5), then chop the square into a fine enough grid such that each rectangle is mapped by F into one of these. Number the grid rectangles lexicographically. Number the neighborhoods such that $\square_i$ lies in U_i. Construct $\tilde{F}$ inductively, one rectangle at a time; show it's unique the same way.



F is a <u>path homotopy</u> if $\forall t$ $F(0,t) = x$ and $F(1,t) = y$ for some $x, y \in B$.

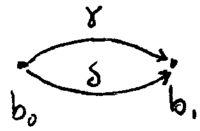


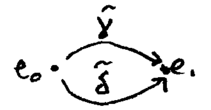
We say F is a path homotopy from γ_0 to γ_1 where

$$\gamma_i(s) = F(s,i)$$

If F is a path homotopy, $F(0,t) = x \ \forall t$
$F(1,t) = y \ \forall t$

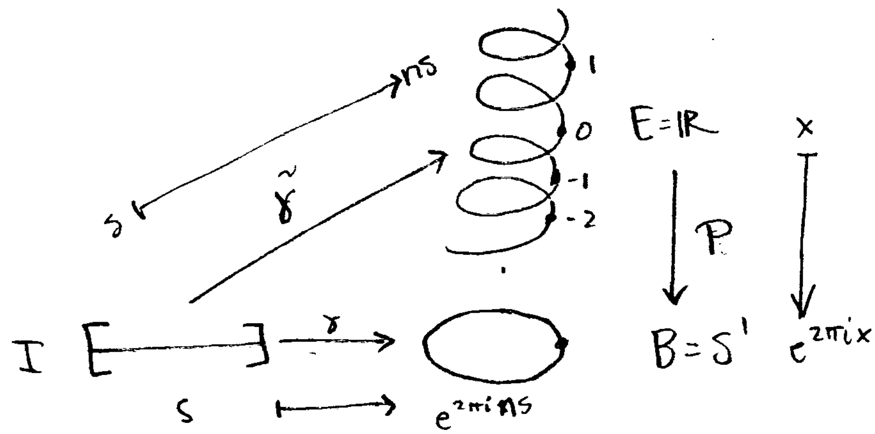
So we have two constant paths, and these are lifted by $\tilde{F}(0,t)$ & $\tilde{F}(1,t)$. Since $p^{-1}(x)$ & $p^{-1}(y)$ are discrete spaces & $\tilde{F}$ continuous, $\tilde{F}(0,t)$ & $\tilde{F}(1,t)$ have to be constant paths. So $\tilde{F}$ is a path homotopy. $\square$

<u>Thm 54.3</u> - Same assumptions. Suppose γ, δ are two paths in the base space 

Then each has unique lifts to paths $\tilde{\gamma}, \tilde{\delta}$ in E starting at e_0 

and if γ is path homotopic to δ then $\tilde{\gamma}$ is path homotopic to $\tilde{\delta}$ and have the same endpoints.

Pf. By previous two lemmas. $\square$



unique once we choose e_0

This allows us to define the lifting map $\phi \; \pi_1(B, b_0) \to p^{-1}(B, b_0) \subseteq E$ as follows: given a loop γ in B, lift it uniquely to a path $\tilde{\gamma}$ in E and take the endpoint $\tilde{\gamma}(1) \in p^{-1}(b_0)$. By Thm 54.3, if γ and δ are path homotopic then $\hat{\gamma}(1) = \tilde{\delta}(1)$, so we can define $\phi([\gamma]) = \tilde{\gamma}(1)$.

Next time:

Thm - Given some assumptions, if E is path connected, then ϕ is onto. If E is simply connected, then ϕ is also 1-1.

Cor - $\mathbb{R}$ is connected and simply connected. $p: \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$ is a covering map and $\phi: \pi_1(S^1, 1) \to p^{-1}(1)$ is 1-1 and onto. So $\phi: \pi_1(S^1, 1) \to \mathbb{Z}$ is a 1-1 correspondence.