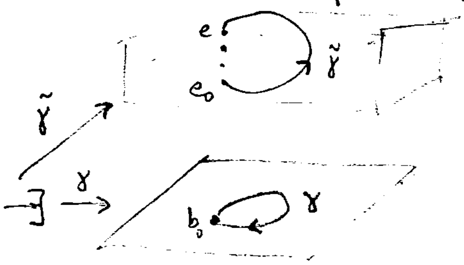
Thm 54.4 - Suppose $p: E \to B$ is a covering map, $b_0 \in B$ and $p(e_0) = b_0$. Then the lifting map $\phi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0) \subset E$ is onto if E is path-connected + also 1-1 if E is also simply connected.

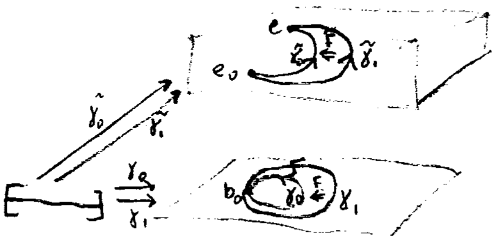
Pf. Suppose E is path connected. To show ϕ is onto, choose $e \in p^{-1}(b_0)$ + find $[\gamma] \in \pi_1(B, b_0)$ st $\phi([\gamma]) = e$



Since E is path connected, $\exists\, \tilde\gamma: [0,1] \to E$ w/ $\tilde\gamma(0) = e_0$, $\tilde\gamma(1) = e$.

If we define $\gamma := p \circ \tilde\gamma$ then $\tilde\gamma$ lifts γ + $\phi([\gamma]) = e = \tilde\gamma(1)$.

Suppose E is simply connected. To show ϕ is 1-1, choose $[\gamma_0], [\gamma_1] \in \pi_1(B, b_0)$ st $\underset{\tilde\gamma_0}{\underset{\shortparallel}{\phi([\gamma_0])}} = \underset{\tilde\gamma_1}{\underset{\shortparallel}{\phi([\gamma_1])}}$ and show $[\gamma_0] = [\gamma_1]$.



To show $[\gamma_0] = [\gamma_1]$, we need to find a homotopy F from γ_0 to γ_1. Since E is simply connected, there's a path homotopy $\tilde\gamma_0 \xrightarrow{\tilde F} \tilde\gamma_1$. Thus $p \circ \tilde F$ is a path homotopy $F: \underset{\gamma_0}{\underset{\shortparallel}{p \circ \tilde\gamma_0}} \longrightarrow \underset{\gamma_1}{\underset{\shortparallel}{p \circ \tilde\gamma_1}}$

$\square$

Thm 54.5 $\quad \pi_1(S^1, 1) \cong \mathbb{Z}$.

Pf. We have a covering map

$$p : \mathbb{R} \to S^1$$
$$x \longmapsto e^{2\pi i x}$$

So we get a lifting map

$$\phi : \pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

if we choose our basepoint to be 1. Since $\mathbb{R}$ is simply-connected, ϕ is 1-1 and onto by 54.4. ϕ is also a group homomorphism:

$$\gamma_n(t) = e^{2\pi i n t} \qquad \tilde{\gamma}_n(t) = nt$$

$$\phi([\gamma_n]) = \tilde{\gamma}_n(1) = n$$

Every elt. of $\pi_1(S^1, 1)$ is one of these $[\gamma_n]$ since ϕ is a bijection, so

$$\phi\left([\gamma_n] * [\gamma_m]\right) = \phi\left([\gamma_{n+m}]\right) = n+m = \phi([\gamma_n]) + \phi([\gamma_m])$$

check this!

$\square$

Applications of $\pi_1(S^1, 1) \cong \mathbb{Z}$:

1. Thm 55.2: there's no retraction $r: D^2 \to S^1$, ie $r(x) = x$ if $x \in S^1$.

Proof. By ex 52.4, if $A \subset X$ is a subspace and $\exists\ r: X \to A$, $* \mapsto *$ a retraction then $\pi_1(A, *) < \pi_1(X, *)$. But

$\pi_1(S^1 \subset D^2, 1) = \mathbb{Z} \not< 0 = \pi_1(D^2, 1)$.

$\square$

2. Thm 55.5: Suppose $\vec{v}$ is a continuous vector field on D^2, i.e. a map $\vec{V}: D^2 \to \mathbb{R}^2$. If $\vec{v}$ is never zero, then $\exists x \in S^1 = \partial D^2$ st. $\vec{v}_x$ points directly outwards and $\exists x'$ st $\vec{v}_{x'}$ points directly inwards.

3. Thm 55.6: (Brover Fixed-point Theorem) For any map $f: D^2 \to D^2$, $\exists x \in D^2$ st $f(x) = x$.