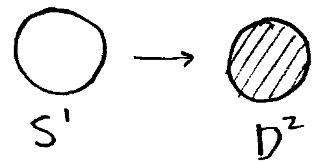
Lemma 55.3 - Given a map $h: S^n \to X$ where

$$S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$$
$$D^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$$

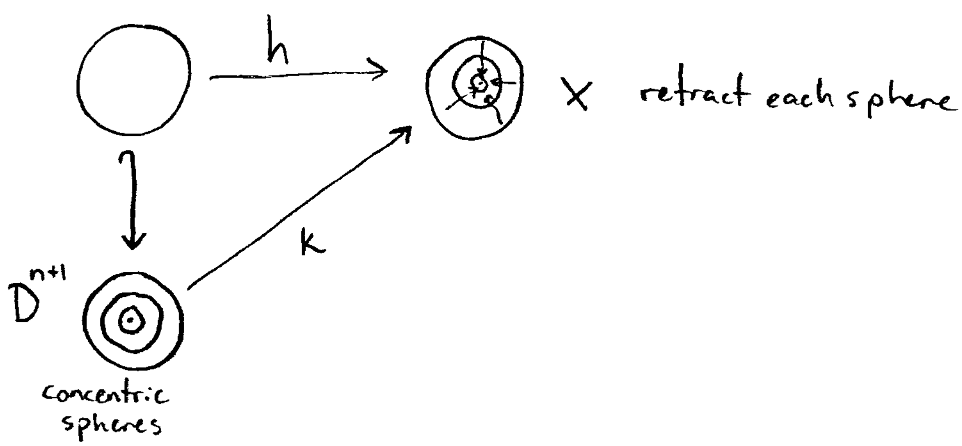


Then h is <u>nulhomotopic</u> to a constant map - if and only if it extends to a map $k: D^{n+1} \to X$ such that

$$\begin{tikzcd} S^n \arrow[r, "h"] \arrow[d, hook, "i"] & X \\ D^{n+1} \arrow[ur, "k"'] & \end{tikzcd}$$

commutes

Sketch of proof (fill in details if you use this to do homework).
Suppose $h: S^n \to X$ is nulhomotopic:



If h is nulhomotopic, we can extend h to k as shown, thinking of D^{n+1} as the union of concentric S^n's. Conversely, given an extension k, define a homotopy

$$F(\underset{\substack{\cap \\ S^n}}{x}, \underset{\substack{\cap \\ [0,1]}}{t}) = k(tx)$$

Note that $F(1,x) = k(x) = h(x)$

but $F(0,x) = k(0x) = k(0)$ is a constant!

So F is a homotopy from h to a constant map.

$\square$

Cor 55.4 - The identity map $1_{S^1}: S^1 \to S^1$ isn't nulhomotopic.

Pf. By Thm 55.3, if 1_{S^1} were nulhomotopic, we could extend it to a map $k: D^2 \to S^1$ that is a retraction since $k|_{S^1} = 1_{S^1}$. But we've shown that no such retraction exists. $\square$

Thm 55.5 - Given a nonvanishing vector field on D^2, i.e. a map

$$\vec{v}: D^2 \to \mathbb{R}^2 - \{0\},$$

then $\exists\, x \in S^1 \subset \mathbb{R}^2$ s.t. $\vec{v}$ points directly outwards, i.e.

$$\vec{v}(x) = \alpha x \quad \text{for } \alpha > 0,$$

and $\exists\, x' \in S^1$ s.t. $\vec{v}$ points directly inwards, i.e.

$$\vec{v}(x) = \alpha x \quad \text{for } \alpha < 0.$$

Proof: Wlog we can assume $\|v(x)\| = 1 \;\forall x \in D^2$ because we can normalize it: $\vec{v}$ continuous implies $\frac{\vec{v}}{\|\vec{v}\|}$ continuous.

Also, if $\vec{v}$ points in- or outwards, so does $\frac{\vec{v}}{\|\vec{v}\|}$.

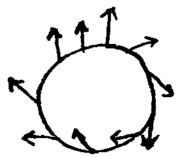
So assume we have a map $\vec{v}: D^2 \to S^1$ s.t. $\vec{v}$ never points inwards; let's get a contradiction.

Let

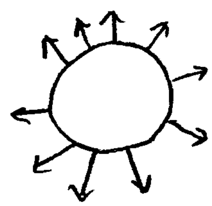
$$\vec{w} = \vec{v}\big|_{S^1} : S^1 \to S^1.$$

Then $\vec{w}$ extends to $\vec{v}$ by definition and by Lem 55.3, $\vec{w}$ must be nulhomotopic.

But $\vec{w}$ is homotopic to 1_{S^1}

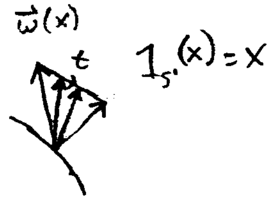


$\vec{w}$ never points toward the center



1_{S^1} points straight out

We have a homotopy



$$F(x,t) = \frac{(1-t)x + t\,\vec{w}(x)}{\|(1-t)x + t\vec{w}(x)\|}$$

Note that the only time this could fail is if $w(x) = \alpha x$ for $\alpha < 0$.

Now since 1_{S^1} is homotopic to $\vec{w}$, which by Cor 55.4 is not nulhomotopic.

So by contradiction, $\exists x \in S^1$ s.t. $\vec{v}(x) = \alpha x$ for $\alpha < 0$.

Since $-v : D^2 \to S^1$ is also a vector field, it too must point directly inwards somewhere on S^1, so v must point directly outwards somewhere.

$\square$

Thm 55.6 (Brouwer Fixed-point Theorem) Any map $f: D^2 \to D^2$ has a fixed point, i.e. $\exists x \in D^2$ s.t. $f(x) = x$.

Pf. By contradiction: suppose $\forall x \in D^2$ $f(x) \neq x$. Then

$$\vec{v} = f(x) - x$$

is a vector field on D^2 that never vanishes. By Thm 55.5, $\vec{v}|_{S^1}$ must point directly outwards, but then $f(x)$ would not be in D^2. Contradiction.

$\square$

In fact, all these results generalize if you replace S^1 by S^n $(n \geq 1)$ and D^2 by D^{n+1}. Some of these are homework problems; they all follow from the fact (which the book gives you without proof) that there's no retraction from D^{n+1} to S^n. Next time, I'll discuss the proof.