We've just derived lots of consequences of the fact that there's no retraction $r: D^2 \to S^1$. In your homework you've seen that we could generalize all these consequences to higher dimensions if we could prove that there's no retraction $r: D^{n+1} \to S^n$. How can we prove this?

HW due next Friday
55.1, 2, 4e, 4f
Read Cor 55.7!

In the case $n=1$ we used

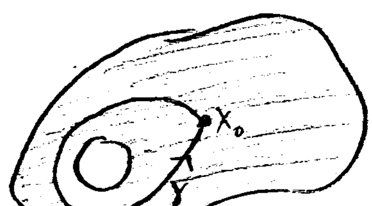
$$\pi_1(S^1, *) = \mathbb{Z} \; \& \; \pi_1(D^2, *) = 0$$

Alas, for $n > 1$ we have

$$\pi_1(S^n, *) = 0 \quad \& \quad \pi_1(D^n, *) = 0$$

So we can't use the same method.

The problem is that π_1 only detects "1-dimensional holes," i.e. holes that you can't contract a 1-dimensional curve around.



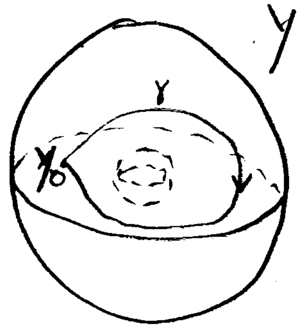
$\gamma: S^1 \to X$ is not nulhomotopic so $\pi_1(X, x_0) \neq 0$

This hole is "caught" by a 1-dimensional lasso, $\gamma: S^1 \to X$, so it's a "1-dimensional hole."

(Note: A loop can be thought of as a path $\tilde{\gamma}: I \to X$ with $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ <u>or</u> as a map $\gamma: S^1 \to X$)

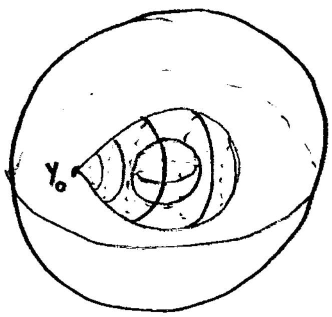
$$\begin{tikzcd} & S^1 = [0,1]/0 \sim 1 \arrow[d, "\gamma"] \\ {[0,1]} \arrow[ur, "\text{quotient map}"] \arrow[r, "\tilde{\gamma}"] & X \end{tikzcd}$$

π_1 fails to detect "2-dimensional holes"



Every loop $\gamma: S^1 \to Y$ is nulhomotopic, so $\pi_1(Y, y_0) = 0$

But we can detect this hole using a surface, i.e. a map $f: S^2 \to Y$!



Not every $f: S^2 \to Y$ is nulhomotopic, so $\pi_2(Y, y_0) \neq 0$.

What's π_2? For any n, we can define $\pi_n(X, x_0)$, where X is a space and $x_0 \in X$, using maps $f: S^n \to X$ s.t. $f(*) = x_0$, where $* \in S^n$ is some chosen point in S^n called the basepoint.

$$\pi_n(X, x_0) = \{\text{based homotopy classes of maps } f: S^n \to X \text{ s.t. } f(*) = x_0\}$$

where a <u>based homotopy</u> is a homotopy

$$F: S^n \times I \to X$$

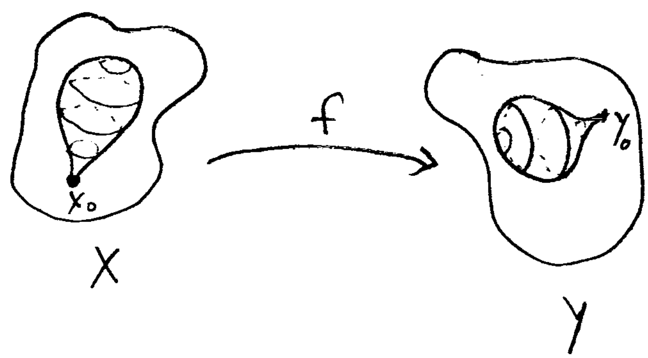
such that $\forall t, F(*, t) = x_0$.

For $n=1$, this really is the fundamental group. (π_0 detects whether a space is connected or not.)

These are some facts about π_n for $n \geq 1$:

1. $\pi_n(X, x_0)$ is a group.
2. Given a map of pointed spaces $f:(X,x_0) \to (Y,y_0)$, i.e. a map such that $f(x_0)=y_0$, there is a homomorphism

$$\pi_n(f): \pi_n(X,x_0) \longrightarrow \pi_n(Y,y_0)$$



3. $\pi_n(fg) = \pi_n(f)\pi_n(g)$
4. $\pi_n(1_X) = 1_{\pi_n(X,x_0)}$

I.e. π_n is a functor from pointed spaces to groups.

Moreover, $\pi_n(S^n, *) = \mathbb{Z}$ (This is hard to show!)

and $\pi_n(D^{n+1}, *) = 0$ (This is easy because D^{n+1} is contractible.)

Given these facts, we can show $\nexists\, r: D^{n+1} \longrightarrow S^n$ s.t. $r|_{S^n} = 1_{S^n}$ in exactly the same way as in the $n=1$ case:

Assume $\exists r$ such that

$$\begin{tikzcd} & D^{n+1} \arrow[dr, "r"] & \\ S^n \arrow[ur, hook, "i"] \arrow[rr, "1_{S^n}"'] & & S^n \end{tikzcd}$$

commutes.

We could hit it with π_n and get groups from our spaces by prop. 1 & homomorphisms from our maps by property 2:

$$\begin{tikzcd} & \pi_n(D^{n+1},*) \arrow[dr, "\pi_n(r)"] & \\ \pi_n(S^n,*) \arrow[ur, hook, "\pi_n(i)"] \arrow[rr, "\pi_n(1_{S^n})"', "= 1_{\pi_n(S^n,*)}" below] & & \pi_n(S_n,*) \end{tikzcd}$$

This new diagram commutes by property 3, and $\pi_n(1_{S^n}) = 1_{\pi_n(S^n,*)}$ by property 4. But we know all these groups:

$$\begin{tikzcd} & 0 \arrow[dr, "\pi_n(r)"] & \\ \mathbb{Z} \arrow[ur, hook, "0"] \arrow[rr, "1"'] & & \mathbb{Z} \end{tikzcd}$$

Contradiction!

Morals:

1. You can turn hard topology problems into easy algebra using functors, e.g. pointed spaces into groups. This is algebraic topology.
2. There are lots of these functors: π_n (fundamental group), H_n (homology groups), H^n (cohomology groups).