Math 205B - Topology

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Theorem 53.1. *The map $p : \mathbb{R} \to S^1$ given by the equation*

$$p(x) = (cos(2\pi x), sin(2\pi x))$$

is a covering map

Proof. First p is continuous since it is the product for continuous functions. Consider the following open subsets of S'.

$$\begin{aligned} U_1 &= \{(cos(2\pi x), sin(2\pi x)) \mid x \in (-\tfrac{1}{4}, \tfrac{1}{4})\} \\ U_2 &= \{(cos(2\pi x), sin(2\pi x)) \mid x \in (\tfrac{1}{4}, \tfrac{3}{4})\} \\ U_3 &= \{(cos(2\pi x), sin(2\pi x)) \mid x \in (0, \tfrac{1}{2})\} \\ U_4 &= \{(cos(2\pi x), sin(2\pi x)) \mid x \in (\tfrac{1}{2}, 1)\} \end{aligned}$$

These open set totally cover S' so we only need to show that each one is evenly covered. Consider the slices of $\mathbb{R}$ to be.

$$\begin{aligned} V_{n_1} &= (n - \tfrac{1}{4}, n + \tfrac{1}{4}) \\ V_{n_2} &= (n + \tfrac{1}{4}, n + \tfrac{3}{4}) \\ V_{n_3} &= (n, n + \tfrac{1}{2}) \\ V_{n_4} &= (n - \tfrac{1}{2}, n) \end{aligned}$$

Where the V_{n_i}'s corresponds to U_i. These evenly cover there corresponding U_i since at least one of $cos(2\pi x)$ or $sin(2\pi x)$ is monotone on these intervals, The end points of the V_{n_i}'s map to the end points of U_i and are thus surjective by the Intermediate Value Theorem, and thus a homeomorphism. □

Theorem 53.4. *If $p : E \to B$ and $p' : E^{'} \to B^{'}$ are covering maps, then*

$$p \times p' : E \times E^{'} \to B \times B^{'}$$

is a covering map.

Proof. Let $b \in B$ and $b' \in B^{'}$, and let U and $U^{'}$ be the evenly covered neighborhoods of b and b'. Also, let $\{V_\alpha\}$ and $\{V^{'}_\beta\}$ be the slices of $p^{-1}(U)$ and $p^{-1}(U^{'})$. Consider the inverse image of $U \times U^{'}$ under $p \times p'$.

$$\begin{aligned}(p \times p')^{-1}(U \times U^{'} &= p^{-1}(U) \times p^{-1}(U^{'}) \\ &= \cup V_\alpha \times \cup V^{'}_\beta \\ &= \cup (V_\alpha \times V^{'}_\beta)\end{aligned}$$

Since each of the families $\{V_\alpha\}$ and $\{V^{'}_\beta\}$ are disjoint and open, then the family $\{V_\alpha \times V^{'}_\beta\}$ is also disjoint and open, and since p and p' are homeomorphisms from the respective slices to the evenly covered neighborhoods, then $p \times p'$ is a homeomorphism from $V_\alpha \times V^{'}_\beta$ to $U \times U^{'}$, and so $p \times p'$ is a covering map. $\square$

Exercise 53.2. Let Y have the discrete topology. Show that if $p : X \times Y \to X$ is projection on to the first coordinate, then p is a covering map.

Proof. First, p is continuous since for any open set $U \in X$, $p^{-1}(U) = \{U \times V \mid V \subseteq Y\}$. Since Y has the discrete topology, then all $V \subseteq Y$ are open, and so $p^{-1}(U)$ is open since it is a union of open set, thus p is continuous. p is surjective since it is a projection.
We will show that X itself is evenly covered to prove p is a covering map. We can write $p^{-1}(X)$ as the disjoint union of open sets $V_y = X \times \{y\}$ for all $y \in Y$. p restricted to V_y is already surjective and continuous, so we only need to show injective and p^{-1} (the canonical embedding) is continuous for each V_y to get our homeomorphism. Let $(x_1, y), (x_2, y) \in X \times Y$. With this, $p(x_1, y_1) = p(x_2, y_2)$ implies $x_1 = x_2$, which gives us $(x_1, y) = (x_2, y)$. Finally for any open set $U \times W \subseteq X \times Y$, we have the pre-image to be the open set U, so p^{-1} is continuous. Thus p restricted to V_y is a homeomorphism, so p is a covering map. $\square$

Exercise 53.4. Let $p : E \to B$ be a covering map; Let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a ***k-fold covering*** of B.

Proof. Since B is connected than the only subsets of B that are both open and closed are B and $\emptyset$. We will show that the set $A_k = \{b \in B \mid p^{-1}(b)\text{has } k \text{ elements}\}$ is non-empty, open, and closed, and thus equal to B.
First, $A \neq \emptyset$ since $b_0 \in A$. Next since p is a covering map, then for the each point $b \in B$, there exist a neighborhood U_b that is evenly covered. If $p^{-1}(b)$ contains k elements, then there are k slices of E that are each homeomorphic to U_b, so each element of U_b has k elements in its pre-image, thus U_b is contained in A_k. This gives us that A_k is the union of these evenly covered neighborhoods, so A_k is open. B is the disjoint union of a family of sets $\{A_j\}$ where each A_j is defined the same way as A_k above. Since each of these is open by the argument above, we have that $B - A_k$ is open, and so A_k is closed. Therefore $B = A_k$. □

Exercise 53.6. Show that the map of Example 3 is a covering map. Generalize to the map $p(z) = z^n$.

Proof. We will generalize since this proof holds for $n = 2$. Let $p(z) = z^n$. p is easily seen to be continuous and surjective. Consider the points of $S^{'}$ to be complex numbers of the form $z = e^{\theta i}$. We will split this into two cases.
Case 1: $z = 1$
For $z = 1$ let $U = S^{'} - \{-1\}$. We will consider the disjoint family of sets in the domain to be $V_j = \{e^{\theta i} \mid \frac{2(j-1)\pi}{n} - \frac{\pi}{n} < \frac{2(j-1)\pi}{n} < \frac{2(j-1)\pi}{n} + \frac{\pi}{n}\}$ for $j = 1, ..., n$. p restricted to V_j is injective since z^n is monotone over an interval smaller than $\frac{2\pi}{n}$. It is surjective since the endpoint each map to -1 in $S^{'}$, so by the Intermediate Value Theorem each point has a pre-image. The inverse $p^{'} = z^{\frac{1}{n}}$ is also continuous, so $V_j \cong U$ for each j.
Case 2: $z \neq 1$
For $z \neq 1$ let $U = S^{'} - 1$. For this we have that $V_j = \{e^{\theta i} \mid \frac{2(j-1)\pi}{n} < \theta < \frac{2j\pi}{n}\}$ for $j = 1, ..., n$. These V_j's are homeomorphic to U by the same argument above.

□