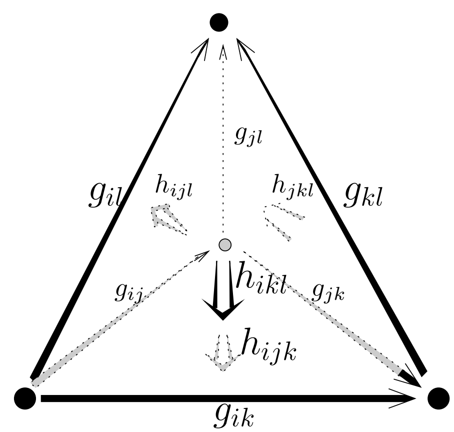
Classifying Spaces For Topological 2-Groups

John Baez and Danny Stevenson

January 7, 2009



for a longer version with references, see:

http://math.ucr.edu/home/baez/barcelona/

A Famous Old Theorem

Here is the result we'd like to categorify:

Thm. Let G be a well-pointed topological group. Let BG, the **classifying space** of G, be the geometric realization of the nerve of G. Then for any paracompact Hausdorff space M, there is a bijection

$$[M, BG] \cong \check{H}^1(M, G)$$

(A topological group G is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)

Topological 2-Groupoids

Defn. A **2-groupoid** is a strict 2-category where all morphisms and 2-morphisms are strictly invertible.

Defn. A **topological 2-groupoid** $\mathcal{G}$ is a 2-groupoid internal to Top.

In other words, $\mathcal{G}$ has:

- a topological space of objects,
- a topological space of morphisms,
- a topological space of 2-morphisms,

and all the 2-groupoid operations are continuous.

Topological 2-Groups

Defn. A **topological 2-group** is a topological 2-groupoid with one object.

So, it has one object: $\bullet$

together with 1-morphisms: $\bullet \xrightarrow{g} \bullet$

and 2-morphisms:
$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "g'"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd}$$

The Čech 2-Groupoid

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space M.

Defn. The **Čech 2-groupoid** $\widehat{\mathcal{U}}$ is the topological 2-groupoid where:

- objects are pairs (x, i) with $x \in U_i$,
- there is a single morphism from (x, i) to (x, j) when $x \in U_i \cap U_j$, and none otherwise,
- there are only identity 2-morphisms.

(This is just a topological groupoid promoted to a 2-groupoid by throwing in identity 2-morphisms.)

Čech Cohomology for 2-Bundles

Defn. A **Čech cocycle** with coefficients in a topological 2-group $\mathcal{G}$ is a continuous weak 2-functor $g\colon \widehat{\mathcal{U}} \to \mathcal{G}$.

Defn. Two Čech cocycles g, g' are **cohomologous** if there is a continuous weak natural isomorphism $f\colon g \Rightarrow g'$.

Defn. Let $\check{H}^1(\mathcal{U}, \mathcal{G})$ be the set of cohomology classes of Čech cocycles. We define the **Čech cohomology** of M with coefficients in $\mathcal{G}$ to be the limit as we refine the cover:

$$\check{H}^1(M, \mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$$

Categorifying That Famous Old Theorem

Thm. Suppose $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\check{H}^1(M,\mathcal{G}) \cong [M, B|N\mathcal{G}|]$$

where the topological group $|N\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $B|N\mathcal{G}|$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group G is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of sets in the cover is contractible.)

How to Build the Classifying Space

First we think of $\mathcal{G}$ as a group in TopGpd and apply the nerve construction:

$$N\colon \mathrm{TopGpd} \to \mathrm{Top}^{\Delta^{\mathrm{op}}}$$

to get a group in simplicial spaces, $N\mathcal{G}$.

Then we use geometric realization:

$$|\cdot|\colon \mathrm{Top}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$$

to get a topological group $|N\mathcal{G}|$.

Then we think of $|N\mathcal{G}|$ as a 1-object topological groupoid, and take the nerve and the geometric realization *of this* to get our space $B|N\mathcal{G}|$.

A Corollary:
Bundles vs. 2-Bundles

Cor. There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over M
- elements of the Čech cohomology $\check{H}^1(M, \mathcal{G})$
- homotopy classes of maps $f\colon M \to B|N\mathcal{G}|$
- elements of the Čech cohomology $\check{H}^1(M, |N\mathcal{G}|)$
- isomorphism classes of principal $|N\mathcal{G}|$-bundles over X.

Another Corollary

For any simply-connected compact simple Lie group G there is a topological 2-group $\mathcal{G}$ called the **string 2-group** of G, such that $|N\mathcal{G}|$ is the 3-connected cover of G.

The homomorphism $|N\mathcal{G}| \xrightarrow{p} G$ gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|N\mathcal{G}|, \mathbb{R})$$

This is onto, with kernel generated by the '2nd Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

So, the real characteristic classes of String(G)-2-bundles are just like those of G-bundles, but with c_2 set to zero!