ASYMPTOTICS OF $10j$ SYMBOLS

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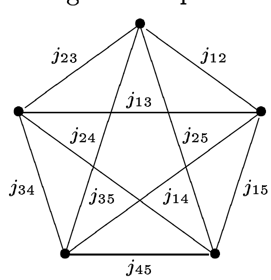
Abstract. The Riemannian $10j$ symbols are spin networks that assign an amplitude to each 4-simplex in the Barrett-Crane model of Riemannian quantum gravity. This amplitude is a function of the areas of the 10 faces of the 4-simplex, and Barrett and Williams have shown that one contribution to its asymptotics comes from the Regge action for all non-degenerate 4-simplices with the specified face areas. However, we show numerically that the dominant contribution comes from degenerate 4-simplices. As a consequence, one can compute the asymptotics of the Riemannian $10j$ symbols by evaluating a 'degenerate spin network', where the rotation group SO(4) is replaced by the Euclidean group of isometries of $\mathbb{R}^3$. We conjecture formulas for the asymptotics of a large class of Riemannian and Lorentzian spin networks in terms of these degenerate spin networks, and check these formulas in some special cases. Among other things, this conjecture implies that the Lorentzian $10j$ symbols are asymptotic to 1/16 times the Riemannian ones.

1. Introduction

In the Ponzano–Regge model of 3-dimensional Riemannian quantum gravity [1], an amplitude is associated with each tetrahedron in a triangulation of spacetime. The amplitude depends on the tetrahedron's six edge lengths, which are assumed to be quantized, taking values proportional to $2j + 1$ where j is a half-integer spin. One can compute this amplitude either by evaluating an SU(2) spin network shaped like a tetrahedron, or by doing an integral. Approximating this integral by the stationary phase method, Ponzano and Regge argued that when all six spins are rescaled by the same factor λ, the $\lambda \to \infty$ asymptotics of the amplitude are given by a simple function of the volume of the tetrahedron and the Regge calculus version of its Einstein action. Nobody has yet succeeded in making their argument rigorous, but Roberts [2, 3] recently proved their asymptotic formula by a different method. This result lays the foundation for a careful study of the relation between the Ponzano–Regge model and classical general relativity in 3 dimensions.

Our concern here is whether a similar result holds for the Barrett–Crane model of *4-dimensional* Riemannian quantum gravity [4]. In this model an amplitude is associated with each 4-simplex in a triangulation of spacetime. This amplitude, known as a $10j$ symbol, is a function of the areas of the 10 triangular faces of the 4-simplex. Each triangle area is proportional to $2j + 1$, where j is a spin labelling the triangle. The amplitude can be computed by evaluating an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ spin network whose edges correspond to the triangles of the 4-simplex:



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There is also an integral formula for the $10j$ symbol [7]. The problem is to understand the asymptotics of the $10j$ symbol as all 10 spins are rescaled by a factor λ and $\lambda \to \infty$.

Barrett and Williams [8] applied a stationary phase approximation to the integral for the $10j$ symbol, focusing attention on stationary phase points corresponding to nondegenerate 4-simplices with the specified face areas. They showed that each such 4-simplex contributes to the $10j$ symbol in a manner that depends on its Regge action. They pointed out the existence of contributions from degenerate 4-simplices, but did not analyse them.

In Section 2 of this paper we begin by applying Barrett and Williams' estimate of the $10j$ symbols to the case where all 10 spins are equal. We find that the contribution of their stationary phase points to the $10j$ symbols is of order $\lambda^{-9/2}$. However, our numerical calculations show that the $10j$ symbols are much larger, of order λ^{-2}. This means we must look elsewhere to explain the asymptotics of the $10j$ symbol.

In Section 3 we analyse the contribution of 'degenerate 4-simplices' to the integral for the $10j$ symbols. They do not correspond to stationary phase points in the integral for the $10j$ symbols; instead, the integrand has a strong *maximum* at these points. We argue that the contribution of a small neighborhood of these points is asymptotically proportional to λ^{-2}. We give a formula expressing the constant of proportionality as an integral over the space of degenerate 4-simplices. We also reduce this to an explicit integral in 5 variables.

In Section 4, we numerically compare these results to the $10j$ symbols as calculated using the algorithm developed by Christensen and Egan [9]. Our formula for the contribution of degenerate 4-simplices closely matches the actual asymptotics of the $10j$ symbols. Thus, even though our argument that these asymptotics are dominated by degenerate 4-simplices is not rigorous, we feel confident that the resulting formula is correct.

In Section 5 we discuss a new sort of spin network, associated to the representation theory of the Euclidean group, which arises naturally in our analysis of the contribution of degenerate 4-simplices. Generalizing our results on the Riemannian $10j$ symbols, we conjecture formulas for the asymptotics of a large class of Riemannian spin networks in terms of these new 'degenerate spin networks'. We verify this conjecture in a number of simple cases.

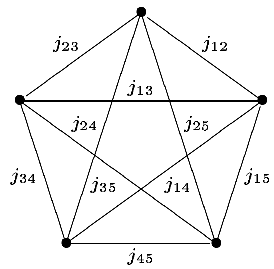
In Section 6 we formulate a similar conjecture for Lorentzian spin networks. Taken with the previous one this conjecture implies that the $\lambda \to \infty$ asymptotics of a Lorentzian spin network in this class are the same, up to a constant, as those of the corresponding Riemannian spin network. For example, as $\lambda \to \infty$, the Lorentzian $10j$ symbol should be asymptotic to $1/16$ times the corresponding Riemannian $10j$ symbol! We conclude by presenting some numerical evidence that this is the case.

2. Stationary Phase Points

The $10j$ symbols can be defined using a Riemannian spin network — also known as a 'balanced' spin network [4] — whose underlying graph is the complete graph on five vertices. The ten edges of the graph are labelled with half-integer spins $j_{kl} = 0, \frac{1}{2}, 1, \ldots$, where k and l refer to the vertices connected by each edge. In this approach, an edge labelled by the spin j corresponds to the representation $j \otimes j$ of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and the $10j$ symbols are computed using the representation theory of this group.

However, to analyse the asymptotics of the $10j$ symbol, it is easier to use the integral formula due to Barrett [7]. This is:

(1)



$$= (-1)^{\sum_{k<l} 2j_{kl}} \int_{(S^3)^5} \prod_{k<l} K^R_{2j_{kl}+1}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2},$$

where S^3 is the unit 3-sphere in $\mathbb{R}^4$ equipped with its usual Lebesgue measure, ϕ_{kl} is the angle between the unit vectors h_k and h_l, and the kernel K^R is given by:

$$K^R_a(\phi) := \frac{\sin a\phi}{\sin \phi}. \tag{2}$$

The normalizing factors in the integral come from the fact that the volume of the unit 3-sphere is $2\pi^2$.

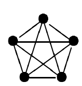
The $10j$ symbol gives the amplitude for a 4-simplex with specified triangle areas. Each vertex of the above graph corresponds to a tetrahedron in this 4-simplex, and each edge of the graph corresponds to the unique triangle shared by two of these tetrahedra. In this picture, the spin j_{kl} determines the area of the triangle shared by the kth and lth tetrahedra. The precise formula for this area is somewhat controversial [10, 11], but given the integral formula for the $10j$ symbols, we find it convenient to assume the area is proportional to $2j_{kl}+1$. Ignoring the constant factor, we thus define triangle areas by:

$$a_{kl} = 2j_{kl} + 1.$$

In what follows, we study the behaviour of the $10j$ symbol as all these triangle areas a_{kl} are multiplied by a large integer λ. This is not the same as multiplying the spins j_{kl} by λ. However, note that as j_{kl} ranges over all spins, a_{kl} ranges over all positive integers. This means that if we multiply the areas a_{kl} by any positive integer λ, we can find new spins J_{kl} corresponding to the new areas by solving $\lambda a_{jk} = 2J_{kl} + 1$.

It is shown in [5] that the integral in (1) is nonnegative. Accordingly, we will concentrate on analysing the asymptotics of the absolute value of the $10j$ symbol, given by the integral alone:

(3)

$\left|\right.$  $\left.\right|$

$$:= \int_{(S^3)^5} \prod_{k<l} K^R_{\lambda a_{kl}}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2},$$

where for simplicity we have left out the spins labelling the spin network edges, which are now J_{kl}.

Barrett and Williams [8] express the numerator in the kernel K^R as a difference of exponentials, which allows them to rewrite the integral in equation (1) as a sum of 2^{10} integrals, each with an integrand consisting of an exponential times the function

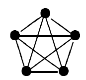
$$f(h_1, \ldots, h_5) = \prod_{k<l} \frac{1}{\sin \phi_{kl}}. \tag{4}$$

This function is unbounded as any of the ϕ_{kl} tend to zero or π, but if the regions of the domain where that occurs are set aside for a separate analysis, the integrals over the remaining region become amenable to stationary phase approximations [14]. The relevant phase is simply:

$$S(h_1, \ldots, h_5) = \sum_{k<l} \lambda a_{kl} \phi_{kl}. \tag{5}$$

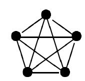
Barrett and Williams show that this function has a stationary point precisely when the ϕ_{kl} are the angles between the outward normals to the tetrahedra of a 4-simplex whose ten faces have areas given by λa_{kl}. In this case the h_k can be interpreted as the outward normals to the five tetrahedra, the ϕ_{kl} are the angles between these normals, and λa_{kl} is the area of the face shared by the tetrahedra numbered k and l. Interpreted this way, S is precisely the Regge action for the 4-simplex.

A geometrical argument in [8] simplifies the original sum of 2^{10} terms, obtaining:

(6) 

$$\left| \cdots \right| = \frac{1}{2^5}\int f(h_1,\ldots,h_5)(e^{iS(h_1,\ldots,h_5)} + e^{-iS(h_1,\ldots,h_5)})\,\frac{dh_1}{2\pi^2}\cdots\frac{dh_5}{2\pi^2}. \tag{6}$$

To carry out a stationary phase approximation of this integral we must note that the stationary phase 'point' determined by the geometry of a 4-simplex is not actually a single point in the 15-dimensional manifold $(S^3)^5$, but rather a whole 6-dimensional submanifold, since the geometry of the 4-simplex is invariant under the 6-parameter rotation group $\mathrm{SO}(4)$. However, the same invariance can be used to remove this complication. Since the functions f and S are invariant under the action of $\mathrm{SO}(4)$, they pass to the quotient space $(S^3)^5/\,\mathrm{SO}(4)$, and we obtain:

(7) 

$$\left| \cdots \right| = \frac{1}{(2\pi)^{10}}\int_{(S^3)^5/\,\mathrm{SO}(4)} f(x)(e^{iS(x)} + e^{-iS(x)})\,d\mu(x), \tag{7}$$

where $d\mu$ is the result of pushing forward Lebesgue measure on $(S^3)^5$ to this quotient space.

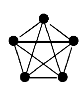
Now, the stationary phase approximation [14] of an n-dimensional integral

$$\int f(x)e^{iS(x)}\,dx$$

is a sum over stationary points x_i of the function S:

$$(2\pi)^{n/2}\sum_i \frac{f(x_i)}{|\det H(x_i)|^{1/2}}\exp\left[iS(x_i) + \frac{i\pi}{4}\sigma(H(x_i))\right],$$

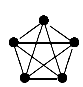
where $H(x_i)$ is the matrix of second partial derivatives of S at the point x_i, and $\sigma(H(x_i))$ is the signature of this matrix, i.e., the number of positive eigenvalues minus the number of negative eigenvalues. This approximation assumes there are finitely many stationary points, all with $\det H(x_i) \neq 0$. Applying this to the case at hand, and neglecting points where some of the angles ϕ_{kl} are 0 or π, we obtain Barrett and Williams' stationary phase approximation of the $10j$ symbols:

(8) 

$$\left| \cdots \right|_{\mathrm{stat}} := \frac{2}{(2\pi)^{11/2}}\sum_i \frac{f(x_i)}{|\det H(x_i)|^{1/2}}\cos\left[S(x_i) + \frac{\pi}{4}\sigma(H(x_i))\right], \tag{8}$$

where the sum is over the 4-simplices x_i with the specified face areas.

We can easily say something about the $\lambda \to \infty$ behavior of this quantity without actually evaluating it. Each stationary point x_i is independent of λ, so $f(x_i)$ will be constant as $\lambda \to \infty$, while $S(x_i)$ will grow linearly, as will its matrix of second derivatives, $H(x_i)$. This is a 9×9 matrix, since $(S^3)^5/\,\mathrm{SO}(4)$ is 9-dimensional, so its determinant will grow as λ^9. It follows that:

(9) 

$$\left| \cdots \right|_{\mathrm{stat}} = O(\lambda^{-9/2}). \tag{9}$$

However, cancellation between different stationary points could in principle make this a misleading overestimate. Thus it seems worthwhile to explicitly evaluate the sum in equation (8), at least in a simple special case.

In what follows we evaluate this sum for the case of a $10j$ symbol with all edges labelled by the same spins. In other words, we calculate the stationary phase approximation of the amplitude for a regular 4-simplex.

The first step, which would be useful more generally, is to find explicit coordinates on the quotient space $(S^3)^5/\,\mathrm{SO}(4)$ and describe the measure $d\mu$ in these terms. To do this, we exploit the fact that any configuration of the five unit vectors h_k can be rotated into one that belongs to a 9-dimensional subspace of the original domain. Specifically, if we express the vectors h_k in polar coordinates

$$h_k = (\cos\psi_k,\ \sin\psi_k\cos\theta_k,\ \sin\psi_k\sin\theta_k\cos\phi_k,\ \sin\psi_k\sin\theta_k\sin\phi_k),$$

where $0 \leq \psi_k, \theta_k \leq \pi$ and $0 \leq \phi_k \leq 2\pi$, any set of h_k can be rotated in such a way that the following restrictions are met:

$$h_1 = (1,\ 0,\ 0,\ 0) \qquad \psi_1 = \theta_1 = \phi_1 = 0 \tag{10a}$$

$$h_2 = (\cos\psi_2,\ \sin\psi_2,\ 0,\ 0) \qquad \theta_2 = \phi_2 = 0 \tag{10b}$$

$$h_3 = (\cos\psi_3,\ \sin\psi_3\cos\theta_3,\ \sin\psi_3\sin\theta_3,\ 0) \qquad \phi_3 = 0. \tag{10c}$$

By means of this 'gauge-fixing' we can use the remaining 9 variables ψ_2, ψ_3, ψ_4, ψ_5, θ_3, θ_4, θ_5, ϕ_4, ϕ_5 as coordinates on the quotient space. To describe the measure $d\mu$ in these coordinates recall that in polar coordinates, Lebesgue measure on the unit 3-sphere is given by

$$dh_k = \sin^2\psi_k\ \sin\theta_k\, d\psi_k\, d\theta_k\, d\phi_k. \tag{11a}$$

Since h_1 is completely fixed we omit dh_1 from the formula for $d\mu$, only inserting a factor $2\pi^2$ due to the volume of the 3-sphere. Since h_2 and h_3 are partially fixed we replace dh_2 and dh_3 by

$$\tilde{dh}_2 = 4\pi\sin^2\psi_2\, d\psi_2 \tag{11b}$$

$$\tilde{dh}_3 = 2\pi\sin^2\psi_3\ \sin\theta_3\, d\psi_3\, d\theta_3. \tag{11c}$$

We thus obtain

$$d\mu = 2\pi^2\, \tilde{dh}_2\, \tilde{dh}_3\, dh_4\, dh_5. \tag{12}$$

Next, we determine the stationary points of the function S in the case where all ten spins are equal, neglecting points where some of the angles ϕ_{kl} are 0 or π — that is, where some of the vectors h_k are parallel or anti-parallel. Recall that these stationary points correspond to nondegenerate 4-simplices having all 10 face areas equal. One obvious candidate is the regular 4-simplex. However, we must rule out the possibility that there are other, *non-regular* 4-simplices for which all the faces have identical areas. Since we are excluding degenerate 4-simplices from the current analysis, we can appeal to a theorem of Bang [15] which states that if all the faces of a non-degenerate tetrahedron have the same area, they are all congruent. It follows that if all ten triangles in a non-degenerate 4-simplex have the same area, they too are all congruent. Now, each of the ten edges of a 4-simplex is shared by three triangles, so we can treat each edge as a triple of congruent line segments that happen to be superimposed, giving a total of 30 in all. If all the triangles were congruent isoceles triangles, each with one side of length L, then ten of these 30 line segments would be of length L. However, there is no way to partition ten line segments into congruent triples. The same argument rules out scalene triangles. So the faces must be equilateral, and the 4-simplex must be regular.

We must therefore find all sets of unit vectors h_k which satisfy the gauge-fixing conditions in equation (10) and form the outward normals of a regular 4-simplex. One choice is:

$$\begin{aligned}
n_1 &= (1,0,0,0)\\
n_2 &= (-\tfrac{1}{4}, \tfrac{\sqrt{15}}{4}, 0, 0)\\
n_3 &= (-\tfrac{1}{4}, -\tfrac{\sqrt{5/3}}{4}, \sqrt{5/6}, 0)\\
n_4 &= (-\tfrac{1}{4}, -\tfrac{\sqrt{5/3}}{4}, -\tfrac{\sqrt{5/6}}{2}, -\tfrac{\sqrt{5/2}}{2})\\
n_5 &= (-\tfrac{1}{4}, -\tfrac{\sqrt{5/3}}{4}, -\tfrac{\sqrt{5/6}}{2}, \tfrac{\sqrt{5/2}}{2}).
\end{aligned}$$

These vectors have mutual dot products of $-\frac{1}{4}$. They represent a single point in our 9-dimensional domain:

$$\psi_2 = \psi_3 = \psi_4 = \psi_5 = \cos^{-1}(-\tfrac{1}{4}) \tag{13a}$$

$$\theta_3 = \theta_4 = \theta_5 = \cos^{-1}(-\tfrac{1}{3}) \tag{13b}$$

$$\phi_4 = -\cos^{-1}(-\tfrac{1}{2}) = -\tfrac{2\pi}{3} \tag{13c}$$

$$\phi_5 = \cos^{-1}(-\tfrac{1}{2}) = \tfrac{2\pi}{3}. \tag{13d}$$

The only other choice comes from interchanging ϕ_4 and ϕ_5. This yields another regular 4-simplex.

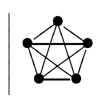
Now we take the phase (5) and specialise to the case where all the spins j_{kl} are equal, say to j. Setting $a = 2j + 1$, and working in our chosen coordinates, we obtain

$$S(\psi_2, \psi_3, \psi_4, \psi_5, \theta_3, \theta_4, \theta_5, \phi_4, \phi_5) = \lambda a \sum_{k<l} \cos^{-1}(h_k \cdot h_l) \tag{14}$$

The partial derivatives of S are all zero at the point described by (13). Symbolic computer calculations show that at this point the matrix of second derivatives of S has 5 positive eigenvalues and 4 negative ones, and a determinant of:

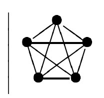
$$\sqrt{\tfrac{5}{3}} \cdot \tfrac{5}{2}\,(\tfrac{2}{3})^{11}\,(\lambda a)^9.$$

Applying this result to (8) at the point described by (13), working in our chosen coordinates, redefining the function f to include the measure (12) as well as the kernel denominators (4), and multiplying by a factor of two to account for the second point where ϕ_4 and ϕ_5 are interchanged, we obtain:



$$\Big|_{\mathrm{stat}} = \frac{48(\frac{3}{5})^{\frac{3}{4}}}{5\pi^{\frac{3}{2}}}(\lambda a)^{-9/2} \cos\left[10\cos^{-1}(-\frac{1}{4})\,\lambda a + \frac{\pi}{4}\right] \tag{15}$$

The above expression consists of an oscillating term times a function proportional to $\lambda^{-9/2}$. In short, the estimate



$$\Big|_{\mathrm{stat}} = O(\lambda^{-9/2})$$

is sharp, at least for the case of 10 equal spins.

However, our numerical calculations of the $10j$ symbols exhibit very different behaviour. Rather than being of order $\lambda^{-9/2}$, they appear to be much larger, of order λ^{-2}. Also, they exhibit no discernable oscillations. In the next section we argue that these results are explained by the contribution of 'degenerate 4-simplices'.

3. Degenerate Points

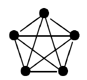
The absolute value of the kernel

$$K^R_a(\phi) = \frac{\sin a\phi}{\sin \phi} \tag{16}$$

has maxima when the angle ϕ equals 0 or π, and these maxima become ever more sharply peaked as $a \to \infty$. This means that as $\lambda \to \infty$, the integrand in equation (3) becomes very large at 'degenerate points', where some of the vectors h_k are either parallel or anti-parallel. We have made a detailed study of the 'fully degenerate' points, where *all* the vectors h_k are either parallel or anti-parallel. It is plausible to expect the contribution from a neighborhood of these points to dominate the integral for the $10j$ symbol, at least asymptotically as $\lambda \to \infty$, since in this region we are integrating a product of kernels, all of which are near their greatest possible absolute value. In the next section we shall present numerical evidence that this is in fact the case.

The value of the kernel is positive at $\phi = 0$, but at $\phi = \pi$ its sign is positive when $2j$ is even and negative otherwise, where $a = 2j + 1$. At first glance, this appears to allow the possibility that parallel and anti-parallel degenerate points might cancel each other for certain values of the spins. In fact, cancellation can occur only when the $10j$ symbol vanishes. The integrand in (1) is the product of ten kernels, so it will be positive when all the h_k are parallel. If h_1 is replaced by its opposite, the four ϕ_{1l} will change from zero to π, leading to an overall sign change of $(-1)^{2(j_{12}+j_{13}+j_{14}+j_{15})}$. However, for the $10j$ symbol to be non-zero, the spins at each vertex must sum to an integer [4]. Therefore, if the $10j$ symbol is non-zero, the sign of the integrand will remain positive. The same argument applies if any subset of the h_k are reversed.

In the integral in equation (3), fully degenerate points occur in 3-dimensional submanifolds of the domain, but we can apply the same gauge-fixing principles as we used to analyse the stationary points, in this case simply fixing $h_1 = (1,\ 0,\ 0,\ 0)$. There are then 16 discrete fully degenerate points in the restricted domain: those where $h_k = \pm h_1$ for $k = 2, \ldots, 5$. Restricting the integral to the vicinity of these points we obtain

(17)

$$\left| \cdots \right|_{\text{deg}} := 16 \int_U \prod_{k<l} K^R_{\lambda a_{kl}}(\phi_{kl})\ \frac{dh_2}{2\pi^2} \cdots \frac{dh_5}{2\pi^2},$$

where U is a small open ball around the point $(h_1, h_1, h_1, h_1) \in (S^3)^4$.

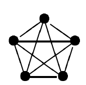
We approximate this quantity by noting that when ϕ is small, the kernel in (16) is close to:

$$K^D_a(\phi) := \frac{\sin a\phi}{\phi} \tag{18}$$

We call this quantity the 'degenerate kernel'. As we shall see in Section 5, this is the analog of the original kernel in a spin-network formalism where the space of constant curvature, S^3, is replaced by three-dimensional Euclidean space, and the group SO(4), the isometry group of S^3, is replaced by the Euclidean group of isometries of $\mathbb{R}^3$.

Similarly, the integral over a small subset of $(S^3)^4$ can be approximated by an integral over a subset of $(\mathbb{R}^3)^4$, and the angle ϕ_{kl} between unit vectors h_k and h_l in S^3 replaced by the Euclidean distance $r_{kl} = |x_k - x_l|$ between vectors x_k and x_l in $\mathbb{R}^3$. In terms of these new Euclidean variables, the restriction $h_1 = (1,\ 0,\ 0,\ 0)$ is replaced by $x_1 = (0,\ 0,\ 0)$.

Thus, when λ is large, we have:

(19)

$$\left| \cdots \right|_{\text{deg}} \approx 16 \int_U \prod_{k<l} K^D_{\lambda a_{kl}}(|x_k - x_l|)\ \frac{dx_2}{2\pi^2} \cdots \frac{dx_5}{2\pi^2},$$

where now U is a small open ball around the origin of $(\mathbb{R}^3)^4$.

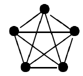
The approximation (19) exhibits very simple scaling behaviour. First, note that the degenerate kernel (18) obeys the identity:

$$\begin{aligned} K^D_{\lambda a}(r) &= \frac{\sin \lambda a r}{r} \\ &= \lambda K^D_a(\lambda r) \end{aligned} \tag{20}$$

This allows the scaling of (19) to be deduced from a linear change of variables, $y_k = \lambda x_k$:

$$\begin{aligned} &16 \int_U \prod_{k<l} K^D_{\lambda a_{kl}}(|x_k - x_l|)\ \frac{dx_2}{2\pi^2} \cdots \frac{dx_5}{2\pi^2} \\ &= 16\lambda^{-2} \int_{\lambda U} \prod_{k<l} K^D_{a_{kl}}(|y_k - y_l|)\ \frac{dy_2}{2\pi^2} \cdots \frac{dy_5}{2\pi^2}, \end{aligned} \tag{21}$$

where λU is the result of rescaling the neighborhood U by a factor of λ. If the integral on the right hand side of this equation converges in the limit $\lambda \to \infty$, we obtain the asymptotic formula:

(22)

$$\left| \cdots \right|_{\mathrm{deg}} \sim 16\lambda^{-2} \int_{(\mathbb{R}^3)^4} \prod_{k<l} K^D_{a_{kl}}(|y_k - y_l|) \, \frac{dy_2}{2\pi^2} \cdots \frac{dy_5}{2\pi^2}.$$

We call this integral the 'degenerate $10j$ symbol'. Apart from the factor of $16\lambda^{-2}$, the integral here can be interpreted as the evaluation of a spin network with edges labelled by unitary irreducible representations of the Euclidean group: the group of isometries of Euclidean 3-space. We discuss this interpretation in more detail in Section 5.

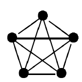
While elegant, the integral in (22) is difficult to compute numerically, very much like the integral for the Lorentzian $10j$ symbol. To obtain a more convenient form for evaluation, we make use of the Kirillov trace formula:

$$\frac{\sin|x|}{|x|} = \frac{1}{4\pi} \int_{S^2} \exp(ix \cdot \xi) \, d\xi,$$

where x is a vector in $\mathbb{R}^3$, and the integral is over the unit 2-sphere with its standard measure. This formula is easily confirmed by choosing spherical coordinates such that the z-axis is parallel to the vector x. For our purposes we shall rewrite it as follows:

$$K^D_a(|x|) = \int_{S(a)} \exp(ix \cdot \xi) \, d\xi, \tag{23}$$

where $S(a)$ is the 2-sphere of radius a embedded in $\mathbb{R}^3$, but where $d\xi$ is the induced Lebesgue measure divided by $4\pi a$, to hide some annoying constants that would otherwise appear in this formula. Using this we can rewrite (22) as:

(24)

$$\begin{aligned} \left| \cdots \right|_{\mathrm{deg}} &\sim \frac{16\lambda^{-2}}{(2\pi^2)^4} \int_{(\mathbb{R}^3)^4} \int_{\mathcal{X}} \exp(i \sum_{k<l} (y_k - y_l) \cdot \xi_{kl}) \, d\xi_{12} \cdots d\xi_{45} \, dy_2 \cdots dy_5 \\ &= \frac{\lambda^{-2}}{\pi^8} \int_{(\mathbb{R}^3)^4} \int_{\mathcal{X}} \exp(i(y_2, y_3, y_4, y_5) \cdot F(\xi_{12}, \ldots \xi_{45})) \, d\xi_{12} \cdots d\xi_{45} \, dy_2 \cdots dy_5 \\ &= \lambda^{-2} 2^{12} \pi^4 \int_{\mathcal{X}} \delta^{12}(F(\xi_{12}, \ldots, \xi_{45})) \, d\xi_{12} \cdots d\xi_{45}, \end{aligned}$$

where

$$\mathcal{X} = \prod_{k<l} S(a_{kl})$$

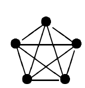
is a Cartesian product of 2-spheres with the measures described above, and $F \colon \mathcal{X} \to \mathbb{R}^{12}$ is defined by:

$$\begin{aligned} F(\xi_{12}, \ldots, \xi_{45}) = (& -\xi_{12} + \xi_{23} + \xi_{24} + \xi_{25}, \\ & -\xi_{13} - \xi_{23} + \xi_{34} + \xi_{35}, \\ & -\xi_{14} - \xi_{24} - \xi_{34} + \xi_{45}, \\ & -\xi_{15} - \xi_{25} - \xi_{35} - \xi_{45}). \end{aligned}$$

If

$$\mathcal{N} = \{\xi \in \mathcal{X} : \; F(\xi) = 0\}$$

then the final integral in (24) will be well-behaved so long as at each point $\xi \in \mathcal{N}$ the differential of F has maximal rank, namely 12. When this is the case $\mathcal{N}$ is an 8-dimensional submanifold of $\mathcal{X}$, and the integral reduces to:

(25)

$$\left| \cdots \right|_{\mathrm{deg}} \sim \lambda^{-2} \, 2^{12} \pi^4 \int_{\mathcal{N}} |J(\xi)|^{-1} \, d\xi$$

Here $d\xi$ is the Lebesgue measure on $\mathcal{N}$ induced by the Riemannian metric on $\mathcal{X}$, but divided by a factor of $\prod_{k<l} 4\pi a_{kl}$, since we have divided the Lebesgue measure on each sphere by a factor of $4\pi a_{kl}$. If we choose local coordinates $(x^1, \dots, x^{20})$ on $\mathcal{X}$ near $\xi \in \mathcal{N}$ such that $x^1, \dots, x^8$ are zero on $\mathcal{N}$, then $|J(\xi)|$ is the Jacobian determinant of F as a function of the remaining 12 variables $x^9, \dots, x^{20}$.

One way to get solutions of $F = 0$ is to start with 4-simplices in $\mathbb{R}^3$: that is, 5-tuples of points in $\mathbb{R}^3$, together with the ten triangles and five tetrahedra determined by these points. Given such a 4-simplex, let ξ_{kl} be the vector that is normal to the triangle shared by the kth and lth tetrahedra, and has length equal to the area of this triangle. As shown in [16], the four vectors ξ_{kl} normal to the triangles in any one tetrahedron must sum to zero, with appropriate signs:

$$\begin{array}{rcl} -\xi_{12} + \xi_{23} + \xi_{24} + \xi_{25} &=& 0 \\ -\xi_{13} - \xi_{23} + \xi_{34} + \xi_{35} &=& 0 \\ -\xi_{14} - \xi_{24} - \xi_{34} + \xi_{45} &=& 0 \\ -\xi_{15} - \xi_{25} - \xi_{35} - \xi_{45} &=& 0 \\ \xi_{12} + \xi_{13} + \xi_{14} + \xi_{15} &=& 0. \end{array} \tag{26}$$

Each vector appears twice in these formulas, with opposite signs, since the outwards-pointing normal to one tetrahedron is the inwards-pointing normal to another. The first four equations say that $F = 0$; the last is an algebraic consequence of the rest. If $|\xi_{kl}| = a_{kl}$, the vectors ξ_{kl} thus determine a point in $\mathcal{N}$.

This suggests that we think of points of $\mathcal{N}$ as 'degenerate 4-simplices'. However, not every point of $\mathcal{N}$ comes from a 5-tuple of points in $\mathbb{R}^3$ this way. To see this, note that two 5-tuples in $\mathbb{R}^3$ determine the same point in $\mathcal{N}$ if they are translates of each other. The space of 5-tuples modulo translation has dimension $3 \times 5 - 3 = 12$, but the space of solutions of equation (26) has dimension $30 - 4 \times 3 = 18$. Thus there are simply not enough 5-tuples of points in $\mathbb{R}^3$ to account for all solutions of equation (26). We shall still call points of $\mathcal{N}$ 'degenerate 4-simplices', because it is a useful heuristic. However, it is important to take this phrase with a grain of salt.

Of course, for the space $\mathcal{N}$ to be nonempty, there must *exist* a solution of $F = 0$. This imposes certain restrictions on the numbers a_{kl}. For example, there will be a solution of $\xi_{12} + \xi_{13} + \xi_{14} + \xi_{15} = 0$ if and only if these 'tetrahedron inequalities' hold:

$$\begin{array}{rcl} a_{12} &\le& a_{13} + a_{14} + a_{15} \\ a_{13} &\le& a_{12} + a_{14} + a_{15} \\ a_{14} &\le& a_{12} + a_{13} + a_{15} \\ a_{15} &\le& a_{12} + a_{13} + a_{14}. \end{array}$$

In general $\mathcal{N}$ will be empty if the four numbers a_{kl} corresponding to the faces of any one tetrahedron violate the tetrahedron inequalities. In this case the degenerate $10j$ symbols vanish. Similarly, the Riemannian $10j$ symbols vanish if the spins j_{kl} violate the tetrahedron inequalities [4], and for the same sort of geometrical reason [16].

For computational purposes it is helpful to rewrite the integral for the degenerate $10j$ symbols in yet another way. To do this, first we note that we can exploit $\mathrm{SO}(3)$ invariance to convert (25) to an integral over the 5-dimensional manifold $\mathcal{N}/\mathrm{SO}(3)$, by rotating the ξ_{kl} in $\mathcal{X}$ so that they lie in the 17-dimensional subspace where ξ_{23} is fixed at $(0,\ 0,\ a_{23})$, and ξ_{34} lies in the $x > 0$ half of the xz-plane. This corresponds to integrating over all possible values for ξ_{23}, inserting a factor equal to the volume of $S(a_{23})$ with our chosen measure, which is a_{23}, and also integrating over all possible azimuthal coordinates for ξ_{34} and inserting a factor of 2π.

In what follows, we will choose coordinates so that the 12×12 matrix $J(\xi)$ whose determinant we require is block diagonal, with two 3×3 blocks and one 6×6 block.

It turns out to be convenient to parameterise ξ_{34}, not by its angle from the z-axis, but by the length

$$s_1 := |\xi_{23} - \xi_{34}|.$$

Since $\xi_{13}+\xi_{23}=\xi_{34}+\xi_{35}$, these four vectors can be positioned to form the sides of a (possibly non-planar) quadrilateral. Then s_1 is the length of one of the diagonals. With the vectors ξ_{23} and ξ_{34} fixed and the lengths of ξ_{13} and ξ_{35} specified, the only remaining freedom this quadrilateral has, if it is to remain closed, is the 'hinge angle', α_1, between the two triangles that meet along the diagonal. The vectors ξ_{13} and ξ_{35} have 4 degrees of freedom in all, and specifying α_1 removes one of them, leaving 3 which break the quadrilateral. With our chosen measure on $\mathcal{N}$, the product of the measure for the coordinates we are integrating over, and the Jacobian determinant for the 3 that break the quadrilateral, is:

$$\frac{1}{(4\pi)^3 a_{23}}.$$

We can treat a second 4-tuple of vectors more or less identically. If the vector ξ_{12} has an azimuthal angle of ϕ, and we define

$$s_2 := |\xi_{23}-\xi_{12}|,$$

then the quadrilateral formed by $\xi_{23},\xi_{12},\xi_{25}$ and ξ_{24} can be assigned a 'hinge angle' of α_2. This specifies all the degrees of freedom that allow this quadrilateral to remain closed. Once again, the product of the measure for the coordinates we are integrating over, and the Jacobian determinant for the 3 that break the quadrilateral, is:

$$\frac{1}{(4\pi)^3 a_{23}}.$$

So far, we have specified 5 degrees of freedom for $\mathcal{N}/\,\mathrm{SO}(3)$: s_1, s_2, α_1, α_2 and ϕ. No further continuous degrees of freedom remain. The three vectors we have yet to specify must form triangles that complete two quadrilaterals with vectors that have already been parameterised, and because these two triangles have a vector in common, there is no 'hinge' freedom left.

Specifically, the three vectors ξ_{14},ξ_{15} and ξ_{45} must form a tetrahedron by fitting over a triangular base that has, as two of its sides, the vectors:

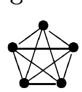
$$\begin{aligned} v &:= \xi_{35}+\xi_{25} \\ w &:= -\xi_{24}-\xi_{34}. \end{aligned}$$

Given their fixed lengths, this determines ξ_{14},ξ_{15} and ξ_{45} completely, apart from the freedom to locate the apex of the tetrahedron on either side of the plane spanned by v and w. This freedom can be accounted for with a factor of 2 in the integral. The final contribution to the Jacobian comes from a 6×6 block involving all the coordinates of ξ_{14},ξ_{15} and ξ_{45}, and with our chosen measure this is:

$$\frac{1}{(4\pi)^3\; 6V(a_{14},a_{15},a_{45},|w|,|v-w|,|v|)},$$

where V is the volume of the tetrahedron as a function of its edge lengths.

Combining these results, we can rewrite (25) as:

(27) 

$$\left|\;\cdot\;\right|_{\mathrm{deg}} \sim \frac{\lambda^{-2}}{96\pi^4 a_{23}}\int_{s_1}\int_{s_2}\int_{\alpha_1}\int_{\alpha_2}\int_{\phi}\frac{ds_1\,ds_2\,d\alpha_1\,d\alpha_2\,d\phi}{V(a_{14},a_{15},a_{45},|w|,|v-w|,|v|)}.$$

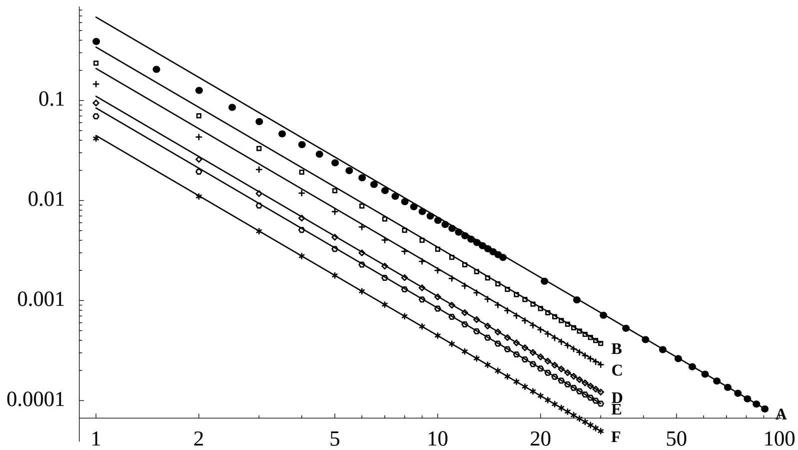
The integrals over the s_i are taken over intervals determined by the four sides of the quadrilaterals for which they are the diagonal lengths, and the angular variables range from 0 to 2π, with the proviso that any part of the domain where the tetrahedron is not geometrically possible must be excluded. In numerical calculations, this can be dealt with by setting the integrand to zero wherever Cayley's determinant formula for the squared volume of the tetrahedron yields a negative value.

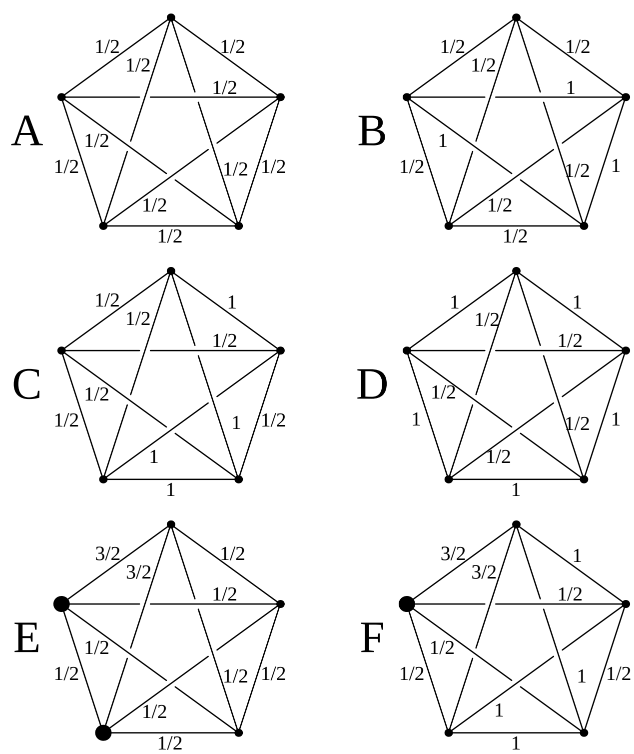
We note that the integrand here is unbounded, and we have not proved that (27) converges, but our numerical calculations suggest that it does.

4. Numerical Data

To test our hypothesis that the asymptotics of 10j symbols are dominated by the contribution of degenerate 4-simplices, we used the algorithm described in [9] to calculate values for several sets of 10j symbols. The figure below shows log-log plots for the absolute values of the Riemannian 10j symbols as a function of λ, where λ is the parameter by which the areas a_{kl} were multiplied. The legend shows the base spins j_{kl}; multiplying a_{kl} by λ was achieved by replacing the j_{kl} with:

$$J_{kl} = \lambda j_{kl} + \frac{\lambda - 1}{2}.$$

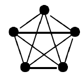




The lines on the plot show the λ^{-2} asymptotic behaviour described by equation (27); the integrals were evaluated numerically with Lepage's VEGAS algorithm [17], and found to have values of 0.680, 0.341, 0.209, 0.110, 0.0841 and 0.0446 respectively.

In summary, our numerical data supports the following conjecture. Let us say that the spins j_{kl} are 'admissible' if for each vertex in the $10j$ symbol, the spins labelling the four incident edges satisfy the tetrahedron inequalities and sum to an integer. Then:

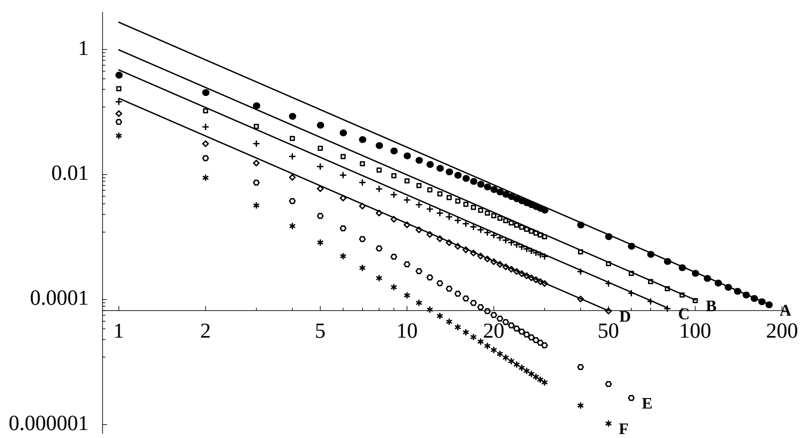
Conjecture 1. *If the ten spins j_{kl} are admissible, the $\lambda \to \infty$ asymptotics of the Riemannian $10j$ symbols are given by:*



$$\left| \cdot \right| \sim 16\lambda^{-2} \int_{(\mathbb{R}^3)^4} \prod_{k<l} K^D_{2j_{kl}+1}(|y_k - y_l|) \, \frac{dy_2}{2\pi^2} \cdots \frac{dy_5}{2\pi^2}.$$

In the next two sections we generalize this conjecture to a large class of Riemannian and Lorentzian spin networks, including the Lorentzian $10j$ symbols.

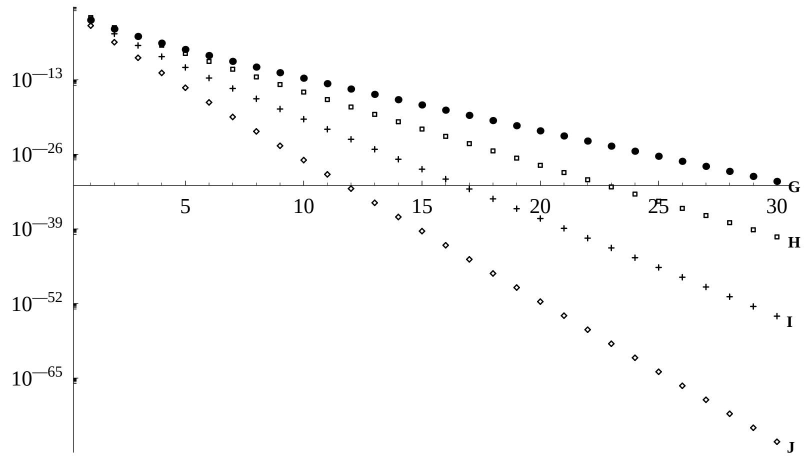
The reader may have wondered why we consider asymptotics of the $10j$ symbols as *areas* are rescaled, instead of *spins*. The reason is that they are much simpler. We also calculated values for sets of $10j$ symbols where the spins j_{kl} were multiplied by λ. The figure below shows log-log plots for the absolute values of the Riemannian $10j$ symbols with spins λj_{kl}, where the j_{kl} match those shown in the legend for the previous figure:

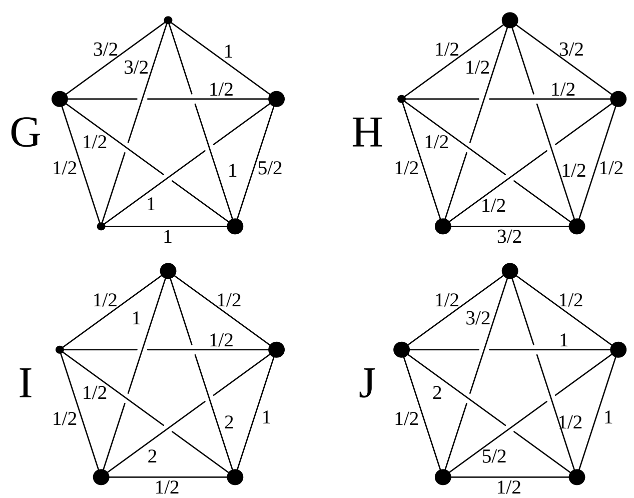


Since the areas $2\lambda j_{kl} + 1$ are asymptotic to a series of values $2\lambda j_{kl}$ that are proportional to λ, the data might be expected to exhibit λ^{-2} scaling. And in fact, in cases A-D, that is exactly what is seen. Furthermore, numerically evaluating the integral (27) with $a_{kl} = 2j_{kl}$ provided the correct coefficients for lines on the plot which are asymptotic to the data; these coefficients were 2.73, 0.987, 0.472 and 0.165 respectively.

For cases E and F, the $10j$ symbols have one or more vertices at the 'border of admissibility': that is, vertices where three of the spins labelling incident edges sum to equal the fourth spin, making one of the tetrahedron inequalities a strict equality. These vertices are marked with heavy dots on the legend. Setting $a_{kl} = 2j_{kl}$ in (27) in these cases yields a coefficient of zero, because the domain for at least one variable of integration, the quadrilateral diagonal length s_1, was reduced to a single point. Empirically, the data here appears to scale as λ^{-3}.

While $10j$ symbols with one or two vertices on the border of admissibility have λ^{-3} asymptotics with spin rescaling, $10j$ symbols with three or more vertices on the border of admissibility appear to decay exponentially as their spins are multiplied by λ, as illustrated in the log-linear plot below:





To prove convergence of the partition function in our newly formulated version of the Barrett–Crane model [6], it will probably be necessary to understand the λ^{-3} and exponential decay of borderline-admissible $10j$ symbols under spin rescaling. However, these are more delicate phenomena than we are prepared to tackle here.

5. Degenerate Spin Networks

Though the Riemannian $10j$ symbols are given by an integral over a product of copies of S^3, we have seen that when we rescale the areas by a large constant λ, this integral is dominated by the contribution of a very small patch of this space, which can be approximated by a product of copies of $\mathbb{R}^3$. Indeed, by a change of variables we can think of the $\lambda \to \infty$ limit as one in which the radius of S^3 approaches infinity, so that it degenerates to Euclidean 3-space.

The Riemannian $10j$ symbols can also be described using the representation theory of SO(4), the isometry group of S^3. This suggests that the $\lambda \to \infty$ asymptotics of the $10j$ symbols can be described in terms of the representation theory of the Euclidean group E(3), the isometry group of $\mathbb{R}^3$.

To do this, we introduce certain spin networks associated to E(3) which we call 'degenerate spin networks'. These are more closely analogous to the Lorentzian spin networks defined in [12] than to the Riemannian spin networks we have been discussing so far, because the edge labels are not restricted to discrete values, and the group representations are infinite-dimensional. However, all three sorts of spin network form part of a unified theory, as outlined below:

geometry	signature	symmetry group	homogeneous space
Riemannian	(++++)	SO(4)	S^3
degenerate	(0+++)	E(3)	$\mathbb{R}^3$
Lorentzian	(−+++)	SO(3,1)	H^3

The rotation group SO(4) and the Lorentz group SO(3,1) consist of linear transformations of $\mathbb{R}^4$ with determinant 1 that preserve metrics of signature (++++) and (−+++), respectively. Similarly, the Euclidean group E(3) is isomorphic to the group of linear transformations of $\mathbb{R}^4$ with determinant 1 that preserve the singly degenerate metric with signature (0+++). The groups SO(4) and SO(3,1) both 'contract' to E(3), meaning that they have it as a limiting case if one forms the isometry group of the metric $\mathrm{diag}(\epsilon, 1, 1, 1)$ and lets $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$, respectively.

This makes it plausible that the asymptotics of not only Riemannian but also *Lorentzian* spin networks can be calculated using degenerate spin networks. In this section we state a precise conjecture along these lines for a large class of Riemannian spin networks, and present some supporting evidence. In Section 6 we do the same for Lorentzian spin networks.

We begin by describing degenerate spin networks and how to evaluate them. The representations $j \otimes j$ labelling edges of a Riemannian spin network are representations not just of Spin(4), but actually of SO(4). As emphasized by Freidel and Krasnov [13], these representations are precisely the eigenspaces of the Laplacian on S^3, which is the homogeneous space $\mathrm{SO}(4)/\mathrm{SO}(3)$. Similarly, the representations labelling edges of a Lorentzian spin network are the eigenspaces of the Laplacian on hyperbolic 3-space, $H^3 = \mathrm{SO}_0(3,1)/\mathrm{SO}(3)$ — here we must take the connected component of the Lorentz group to get just one sheet of the hyperboloid. Following this pattern, the representations labelling edges of a degenerate spin network should be the eigenspaces of the Laplacian on $\mathbb{R}^3 = \mathrm{E}(3)/\mathrm{SO}(3)$.

In fact, there is one representation of this sort for each positive real number a. Any complex function on $\mathbb{R}^3$ with $\nabla^2 f = -a^2 f$ can be written as:

$$f(x) = \int_{\xi \in S(a)} \hat{f}(\xi)\ \exp(i\xi \cdot x)\ d\xi, \tag{28}$$

where $S(a)$ is the 2-sphere of radius a centered at the origin of $\mathbb{R}^3$, and $d\xi$ is the induced Lebesgue measure divided by $4\pi a$. Defining an inner product on these solutions by

$$\langle f, g\rangle = \int_{\xi \in S(a)} \overline{\hat{f}(\xi)}\ \hat{g}(\xi)\ d\xi,$$

we form a Hilbert space $\mathcal{H}_a$ consisting of all solutions f with $\langle f, f\rangle < \infty$. This Hilbert space becomes a representation of the Euclidean group where each group element g acts via $(gf)(x) = f(g^{-1}x)$. In fact, this representation is unitary and irreducible. The functions $\exp(i\xi \cdot x)$ form a 'basis', in the sense that any element of $\mathcal{H}_a$ can be expressed as in equation (28) for a unique square-integrable function $\hat{f}$ on the sphere.

We define a 'degenerate spin network' to be a directed graph with each edge e labelled by a representation of this form, or equivalently, a positive number a_e. An intertwiner between tensor products of these representations can be defined at each vertex by taking the product of functions from the representations labelling the *incoming* edges, multiplying it by the product of complex conjugates of functions from representations labelling the *outgoing* edges, and integrating the result over $\mathbb{R}^3$. Given this, the standard way to evaluate such a spin network [18] would be to take a 'trace': that is, integrate over a basis label $\xi_e \in S(a_e)$ for each edge of the graph. The result would be:

$$\int_{\prod_{v\in V}\mathbb{R}^3} \prod_{e\in E}\left[\int_{S(a_e)} \exp(i(x_{s(e)} - x_{t(e)})\cdot \xi_e)\; d\xi_e\right] \prod_{v\in V} \frac{dx_v}{2\pi^2}.$$

Here E denotes the set of edges of the graph, V denotes its set of vertices, and the vertices $s(e)$ and $t(e)$ are the source and target of the edge e.

However, just as in the Lorentzian case [12], this gives a divergent integral, because the integrand is invariant when we simultaneously translate all the vectors $x_v \in \mathbb{R}^3$ by the same amount. More generally, if the underlying graph of our spin network consists of several connected components, and we translate the vectors x_v where v lies in any one component, the integrand does not change. To keep things simple, let us consider only spin networks whose underlying graph is *connected*. In this case we can sometimes obtain a well-defined integral by 'gauge-fixing' one of the vectors rather than integrating over it: that is, setting $x_{v_1} = x \in \mathbb{R}^3$ for some vertex $v_1 \in V$. If we let $V' = V - \{v_1\}$, this gives the following formula for evaluating a degenerate spin network with edges labelled by the numbers a_e:

$$I^D(a) = \int_{\prod_{v\in V'}\mathbb{R}^3} \prod_{e\in E}\left[\int_{S(a_e)} \exp(i(x_{s(e)} - x_{t(e)})\cdot \xi_e)\; d\xi_e\right] \prod_{v\in V'} \frac{dx_v}{2\pi^2}. \tag{29}$$

As in the Lorentzian case [19], one can show that if this integral converges, the result does not depend on our choice of the special vertex v_1 or the point $x \in \mathbb{R}^3$. Assuming the integral does converge, we can use the Kirillov trace formula (23) to reexpress it as:

$$I^D(a) = \int_{\prod_{v\in V'}\mathbb{R}^3} \prod_{e\in E}\left[K^D_{a_e}(|x_{s(e)} - x_{t(e)}|)\right] \prod_{v\in V'} \frac{dx_v}{2\pi^2}. \tag{30}$$

When the evaluation of a degenerate spin network converges, it always obeys a very simple scaling law as we multiply all the edge labels a_e by the same constant λ. Using the scaling property of the degenerate kernel noted in equation (20), we find that

$$I^D(\lambda a) = \lambda^{|E|-3(|V|-1)}\; I^D(a), \tag{31}$$

where $|E|$ is the number of edges in the underlying graph and $|V|$ is the number of vertices.

We have already argued that the asymptotics of the Riemannian $10j$ symbols are governed by the corresponding degenerate spin network. We can generalize this argument as follows. Fix a connected graph. If we label each edge e by a positive integer a_e — or equivalently a spin j_e with $a_e = 2j_e + 1$ — we obtain a Riemannian spin network, whose evaluation we define by:

$$I^R(a) = \int_{\prod_{v\in V'} S^3} \prod_{e\in E} K^R_{a_e}(d(x_{s(e)}, x_{t(e)})) \prod_{v\in V'} \frac{dx_v}{2\pi^2}. \tag{32}$$

Here $d(x, y)$ is the distance between points x, y in the unit 3-sphere in $\mathbb{R}^4$ as measured by the induced Riemannian metric. This formula is equivalent to the standard integral formula [20], except that we have omitted the usual signs in order to simplify the relationship to degenerate spin networks. Fixing a small open ball U around some point $(x, \ldots, x) \in \prod_{v\in V'} S^3$ we define the 'degenerate contribution' to this integral to be:

$$I^R_{\mathrm{deg}}(a) = 2^{|V|-1} \int_U \prod_{e\in E} K^R_{a_e}(d(x_{s(e)}, x_{t(e)})) \prod_{v\in V'} \frac{dx_v}{2\pi^2}. \tag{33}$$

Just as we included a factor of 16 in equation (17), here we include a factor of $2^{|V|-1}$ to take into account the contribution of anti-parallel degenerate points; as before there is no cancellation between these if the spins labelling edges incident to each vertex sum to an integer, as they must for the spin network to have a nonzero value. Using the same nonrigorous argument as in Section 3, we see that as $\lambda \to \infty$,

$$\begin{aligned} I^R_{\text{deg}}(\lambda a) &\sim 2^{|V|-1} \int_U \prod_{e\in E} K^D_{\lambda a_e}(|x_{s(e)} - x_{t(e)}|) \prod_{v\in V'} \frac{dx_v}{2\pi^2} \\ &= 2^{|V|-1}\, \lambda^{|E|-3(|V|-1)} \int_{\lambda U} \prod_{e\in E} K^D_{a_e}(|y_{s(e)} - y_{t(e)}|) \prod_{v\in V'} \frac{dy_v}{2\pi^2} \\ &\sim 2^{|V|-1}\, \lambda^{|E|-3(|V|-1)}\, I^D(a) \\ &= 2^{|V|-1}\, I^D(\lambda a), \end{aligned} \tag{34}$$

where U now denotes an open ball around the origin of $\prod_{v\in V'} \mathbb{R}^3$, and we made the change of variables $y_e = \lambda x_e$.

In short, this argument suggests that the asymptotics of the degenerate contribution to the value of a Riemannian spin network are proportional to those of the corresponding degenerate spin network:

$$I^R_{\text{deg}}(\lambda a) \sim 2^{|V|-1}\, I^D(\lambda a),$$

and we know the latter are very simple:

$$I^D(\lambda a) = \lambda^{|E|-3(|V|-1)}\, I^D(a).$$

This is particularly interesting when we also have

$$I^R(\lambda a) \sim I^R_{\text{deg}}(\lambda a),$$

because then we can compute the asymptotics of a Riemannian spin network by evaluating a degenerate spin network:

$$I^R(\lambda a) \sim 2^{|V|-1}\, \lambda^{|E|-3(|V|-1)}\, I^D(a).$$

When can we expect this to occur? Clearly we should at least demand that the degenerate contribution outweigh the contribution of stationary phase points. A simple power-counting argument as in Section 2 suggests that the contribution of stationary phase points is of order:

$$I^R_{\text{stat}}(\lambda a) = \begin{cases} O(\lambda^{-\frac{3}{2}|V|+3}) & |V| > 2 \\ O(\lambda^{-\frac{1}{2}}) & |V| = 2 \\ O(\lambda) & |V| = 1, \end{cases} \tag{35}$$

where the graphs with one or two vertices are different because there is less need for 'gauge-fixing'. Comparing these asymptotics to those of the degenerate contribution, we can formulate the following:

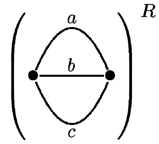
Conjecture 2. *Given a connected graph with more than two vertices and $|E| > \frac{3}{2}|V|$, or two vertices and $|E| > 2$, or one vertex and $|E| > 1$, as $\lambda \to \infty$ we have*

$$I^R(\lambda a) \sim 2^{|V|-1}\, \lambda^{|E|-3(|V|-1)}\, I^D(a)$$

as long as the integral defining $I^D(a)$ converges and the spins j_e labelling the edges incident to each vertex are admissible.

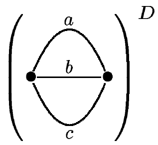
Here we say the spins labelling the edges incident to some vertex are 'admissible' if they sum to an integer and each is less than or equal to the sum of the rest. We do not yet have general criteria for when the integrals associated to Euclidean spin networks converge, and as we shall see, the relevant theorems are bound to be a bit different than in the Lorentzian case [19].

The simplest test of this conjecture is the 'theta network', with two vertices joined by three edges, labelled by positive integers a, b, and c. When the corresponding spins are admissible, the Riemannian theta network evaluates to:

(36) 

$$= 1.$$

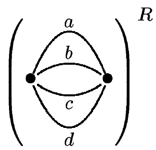
The degenerate theta network can also be explicitly evaluated; assuming without loss of generality that $a \le b \le c$:

(37) 

$$= \begin{cases} 0 & c > a+b \\ \frac{1}{4} & c = a+b \\ \frac{1}{2} & c < a+b. \end{cases}$$

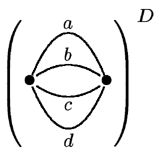
Since the Riemannian network's spins are admissible, the third inequality must hold for the corresponding areas a, b, c. Thus in this case the conjecture gives an *exact* formula for the Riemannian spin network.

The next simplest case is the '$4j$ symbol': the spin network with two vertices joined by four edges, labelled by positive integers a, b, c and d. Without loss of generality let us assume $a \le b \le c \le d$. As noted in a previous paper in this series [6], the Riemannian $4j$ symbol counts the dimension of a space of SU(2) intertwiners. Using this it follows that:

(38) 

$$= \begin{cases} 0 & b+c \le d-a \\ \frac{1}{2}(a+b+c-d) & d-a \le b+c < d+a \\ a & d+a \le b+c. \end{cases}$$

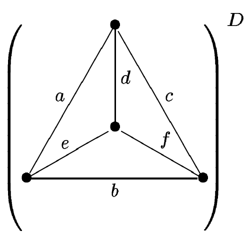
The corresponding degenerate spin network evaluates to:

(39) 

$$= \begin{cases} 0 & b+c \le d-a \\ \frac{1}{4}(a+b+c-d) & d-a \le b+c < d+a \\ \frac{1}{2}a & d+a \le b+c, \end{cases}$$

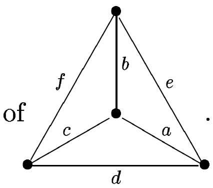
so the conjecture is again exact.

An interesting check on our hypotheses is the tetrahedral spin network. This has four vertices and $|E| = \frac{3}{2}|V|$, so the hypotheses of Conjecture 2 do not apply: we expect the stationary phase contribution to the Riemannian tetrahedral network to be comparable to the degenerate contribution. The degenerate tetrahedral network evaluates to:

(40) 

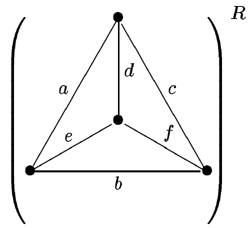
$$= \frac{1}{24\pi V(a,b,c,d,e,f)},$$

where $V(a,b,c,d,e,f)$ is defined as the volume of the tetrahedron dual to the tetrahedral network. Each triangle in this dual tetrahedron corresponds to a vertex of the tetrahedral network, and the three sides of the triangle have lengths equal to the labels on the three edges incident to the network vertex:

(41) $$V(a,b,c,d,e,f) = \text{the volume of}$$ 

.

On the other hand, the Riemannian tetrahedral network evaluates to the square of the SU(2) tetrahedral network, the basic building-block of the Ponzano–Regge model. Thanks to the calculation of Ponzano and Regge [1], later made rigorous by Roberts [2], this means that:

(42) 

$$\sim \frac{\cos^2(S+\frac{\pi}{4})}{12\pi V(\frac{a}{2},\frac{b}{2},\frac{c}{2},\frac{d}{2},\frac{e}{2},\frac{f}{2})}$$

$$= \frac{1+\cos 2(S+\frac{\pi}{4})}{24\pi V(\frac{a}{2},\frac{b}{2},\frac{c}{2},\frac{d}{2},\frac{e}{2},\frac{f}{2})}.$$

Here we are dealing with a dual tetrahedron whose edge lengths are $\frac{1}{2}a_{kl}$, where a_{kl} ranges over a,b,c,d,e,f as $1 \le k < l \le 4$. This tetrahedron has Regge action

$$S = \sum_{1\le k<l\le 4} \tfrac{1}{2} a_{kl}\theta_{kl},$$

where θ_{kl} are the corresponding dihedral angles. Thus it appears that the asymptotics of the Riemannian tetrahedral network are a sum of two parts: a part equal to 8 times the degenerate tetrahedral network, and an oscillatory part coming from the stationary phase points.

When the edge lengths of the dual tetrahedron are such that it cannot exist in Euclidean space, the degenerate tetrahedral network evaluates to zero, and the Riemannian network obeys different asymptotics in which its evaluation exponentially decays with increasing spin.

6. Lorentzian Spin Networks

Now let us turn to Lorentzian spin networks [12, 19]. Each positive real number determines a unitary irreducible representation of the connnected Lorentz group $SO_0(3,1)$ corresponding to an eigenspace of the Laplacian on H^3; however, we prefer to describe the evaluation using an integral formula. Fixing a connected graph with vertex set V and edge set E, and labelling each edge e by a positive real number a_e, we obtain a so-called 'Lorentzian spin network', whose evaluation is given by:

(43) $$I^L(a) = \int_{\prod_{v\in V'} H^3} \prod_{e\in E} K^L_{a_e}(d(x_{s(e)}, x_{t(e)})) \prod_{v\in V'} \frac{dx_v}{2\pi^2}.$$

As with degenerate spin networks, we have chosen a vertex v_1, set $V' = V - \{v_1\}$, and let x_{v_1} be any fixed point in H^3. Here H^3 is hyperbolic 3-space, i.e., the submanifold of Minkowski spacetime given by:

$$H^3 = \{t^2 - x^2 - y^2 - z^2 = 1,\ t > 0\}$$

with its induced Riemannian metric. We define the Lorentzian kernel K^L by:

(44) $$K^L_a(\phi) := \frac{\sin a\phi}{\sinh \phi}.$$

We warn the reader that this convention differs from that of most previous papers [5, 12, 19], which include a factor of a in the denominator. Including that factor would divide any Lorentzian spin network by the product of its edge labels, so for example, it would divide the asymptotics of the Lorentzian $10j$ symbols as defined here by a factor of λ^{10}.

The same line of argument by which we arrived at our conjecture concerning asymptotics of Riemannian spin networks applies to Lorentzian ones. There are only two important differences. First, no factor of $2^{|V|-1}$ appears, since there are no 'antipodal points' in hyperbolic space. Second, one can show that there are no stationary phase points in the Lorentzian integral; dilating a configuration of points in H^3 always changes the associated phase. Optimistically assuming this means that the asymptotics are always given by the contribution of degenerate points, we obtain:

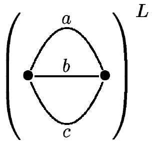
Conjecture 3. *Given any connected graph, as* $\lambda \to \infty$ *we have*

$$I^L(\lambda a) \sim \lambda^{|E|-3(|V|-1)}\, I^D(a)$$

as long as the integral defining $I^D(a)$ *converges and the positive numbers* a_e *labelling edges incident to each vertex are admissible.*

Here we say the positive numbers labelling the edges incident to some vertex are 'admissible' if each is strictly less than the sum of the rest.

Again the simplest test of this conjecture is the Lorentzian theta network. Translating their result into our notation, a calculation of Barrett and Crane [12] shows that for any $a, b, c > 0$,

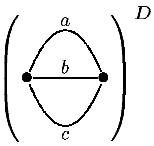


$$= \frac{1}{4}\left[f(-a+b+c) + f(a-b+c) + f(a+b-c) - f(a+b+c)\right],$$

where

$$f(k) = \tanh(\frac{\pi}{2}k).$$

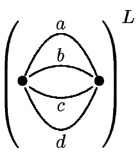
As the conjecture predicts, the asymptotics of this match those of the degenerate theta network, which are given by:



$$= \frac{1}{4}\left[\mathrm{sign}(-a+b+c) + \mathrm{sign}(a-b+c) + \mathrm{sign}(a+b-c) - \mathrm{sign}(a+b+c)\right].$$

It is worth noting that while the integral for the Lorentzian theta network converges even when we take the absolute value of the integrand, this fails for the degenerate theta network. This makes it more challenging to find criteria for convergence of degenerate spin networks, since we cannot simply mimic the theory that applies in the Lorentzian case [19].

Barrett and Crane also worked out the Lorentzian $4j$ symbols, obtaining:

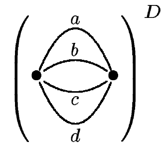


$$= \frac{1}{4}[g(-a+b+c+d) + g(a-b+c+d) + g(a+b-c+d) + g(a+b+c-d)$$
$$-g(a+b-c-d) - g(a-b+c-d) - g(a-b-c+d) - g(a+b+c+d)],$$

where

$$g(k) = \frac{k}{2}\coth(\frac{\pi}{2}k).$$

¿From equation (39) one can show:



$$=$$

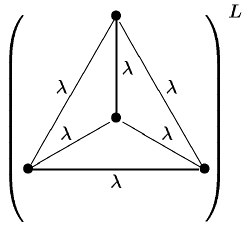
$$\frac{1}{4}[h(-a+b+c+d)+h(a-b+c+d)+h(a+b-c+d)+h(a+b+c-d) \\ -h(a+b-c-d)-h(a-b+c-d)-h(a-b-c+d)-h(a+b+c+d)],$$

where

$$h(k)=\frac{k}{2}\operatorname{sign}(k).$$

Thus the conjecture is also confirmed in this case.

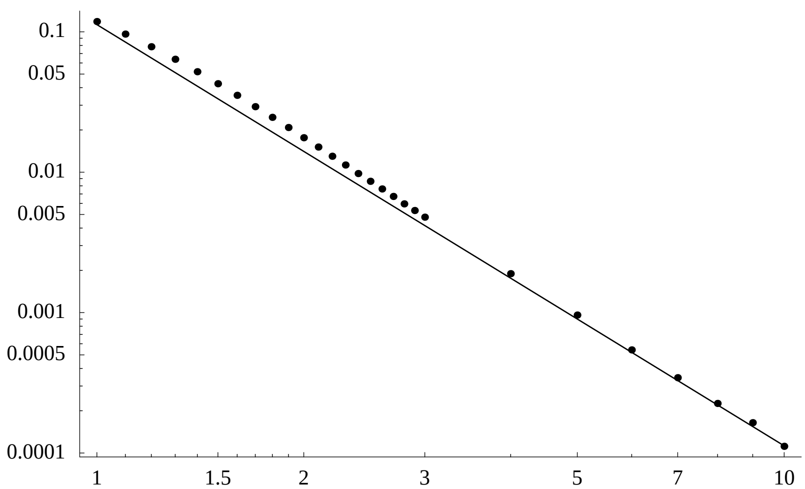
Next, consider the tetrahedral spin network. In the Riemannian case, Conjecture 2 did not apply to this spin network because we expected an equally important contribution due to stationary phase points. In the Lorentzian case, however, there are no stationary phase points, so Conjecture 3 predicts that:



$$\sim \frac{1}{24\pi V(\lambda,\lambda,\lambda,\lambda,\lambda,\lambda)}$$

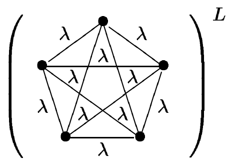
$$=\frac{\sqrt{2}}{4\pi}\lambda^{-3}.$$

Below is a log-log plot comparing this prediction to numerical data. The horizontal axis in this graph represents λ, while the vertical axis represents the value of the tetrahedral spin network:



The most interesting test of Conjecture 3 is the $10j$ symbol. If the conjecture is true, the Lorentzian $10j$ symbol should be asymptotic to the degenerate $10j$ symbol, and therefore asymptotic to $1/16$ times the Riemannian $10j$ symbol. Since the Riemannian $10j$ symbol is positive [5], this in turn would imply that the Lorentzian and degenerate $10j$ symbols are positive in the $\lambda \to \infty$ limit.

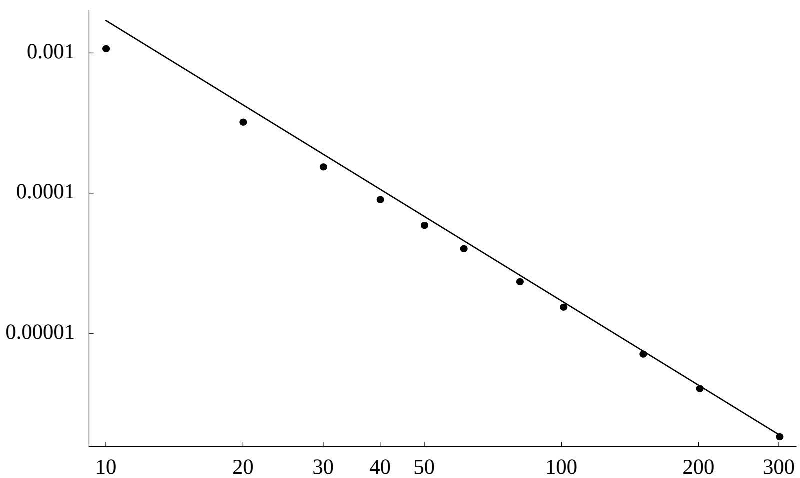
It is difficult to compute the Lorentzian $10j$ symbol, but we have numerically checked the conjecture in the special case where all the edges labelled by the same number λ. In this case the conjecture says that:



$$:= \int_{(H^3)^4} \prod_{k<l} K_\lambda^L(\phi_{kl}) \, \frac{dx_2}{2\pi^2} \cdots \frac{dx_5}{2\pi^2}$$

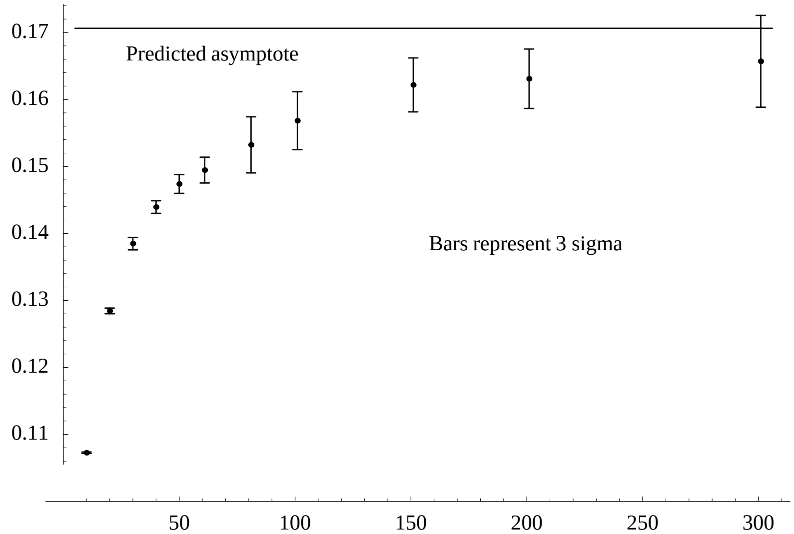
$$\sim .1706\,\lambda^{-2},$$

where of course the constant is not exact. Below we show a log-log plot comparing this prediction to data points obtained by computing the Lorentzian $10j$ symbols numerically. The horizontal axis represents λ, while the vertical axis represents the value of the $10j$ symbol.



Here we computed the Lorentzian $10j$ symbols by applying the VEGAS algorithm to evaluate the integral above using polar coordinates on H^3. The integral was reduced from 12 to 9 dimensions by exploiting the $\mathrm{SO}_0(3,1)$ invariance; the infinite domain was made compact by replacing the radial coordinate for each point, r_i, with a new variable $t_i = r_i/(1+r_i)$; and the domain was further reduced by exploiting a 24-fold symmetry present in the regular case. The large dimension and oscillatory nature of this integral make these calculations extremely computationally intensive.

The graph below shows the same Lorentzian $10j$ symbols multiplied by λ^2, plotted on a linear scale to give a clearer picture of the rate of convergence towards the asymptote. The error bars are three times the standard deviation computed by the VEGAS algorithm. The coordinate system we used yielded the lowest standard deviations of several we tried, but there is no reason to believe that the estimates of the integral are drawn from a normal distribution, so this data should be treated with caution.



7. Conclusions

It appears that the asymptotics of both Riemannian and Lorentzian $10j$ symbols can be computed in terms of degenerate spin networks. For the mathematician, this claim still requires proof. Indeed, there is even an issue of rigor concerning our argument that degenerate spin networks asymptotically describe the contribution of fully degenerate points, since we have not proved that the limit

$$\lim_{\lambda\to\infty} \int_{\lambda U} \prod_{e\in E} K^D_{a_e}(|y_{s(e)} - y_{t(e)}|) \prod_{v\in V'} \frac{dy_v}{2\pi^2}$$

exists. The ambitious mathematician could also try to prove more general versions of our conjectures, in which all the areas a_e approach infinity, but not in fixed proportion to one another.

For the physicist, however, a more pressing question is: *what do these results imply for the physics of the Barrett–Crane model?* On the one hand, it is unsettling that the simple asymptotic behavior of the Ponzano–Regge model is not found here. The way degenerate geometries govern the asymptotics of the $10j$ symbols raises the possibility that in the limit of large spins, the Barrett–Crane model reduces to a theory of degenerate metrics. However, it is important to bear in mind that the physics of the Barrett–Crane model may not be controlled by the large-spin behavior of the $10j$ symbols — though certainly this affects the convergence of the partition function [6]. Indeed, there are still major open questions about the right way to extract physics from spin foam models, and we hope that our work spurs on research on this subject.

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