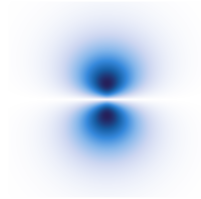
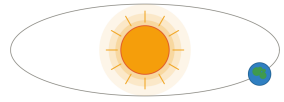
Second Quantization of the Kepler Problem

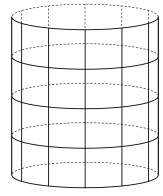
the atom,



the solar system,



and the universe



John Baez

Integrable Systems and Mathematical Physics Seminar

September 22, 2026

The classical particle moving in an elliptical orbit in an inverse square force $\cong$
the classical relativistic free massless spin-0 particle in $\mathbb{R} \times S^3$

$\Downarrow$ quantization

The nonrelativistic hydrogen atom with spin-0 electron $\cong$
the quantum relativistic free massless spin-0 particle in $\mathbb{R} \times S^3$

$\Downarrow$ including spin

The nonrelativistic hydrogen atom with spin-$\frac{1}{2}$ electron $\cong$
the quantum relativistic free massless spin-$\frac{1}{2}$ particle in $\mathbb{R} \times S^3$

$\Downarrow$ second quantization

The atom with many noninteracting spin-$\frac{1}{2}$ electrons $\cong$
the massless spin-$\frac{1}{2}$ relativistic free quantum field in $\mathbb{R} \times S^3$

The classical Kepler problem

If we work in units where the inverse square force law says

$$\ddot{\mathbf{q}} = -\frac{\mathbf{q}}{q^3}$$

and define momentum by $\mathbf{p} = \dot{\mathbf{q}}$, then not only are energy

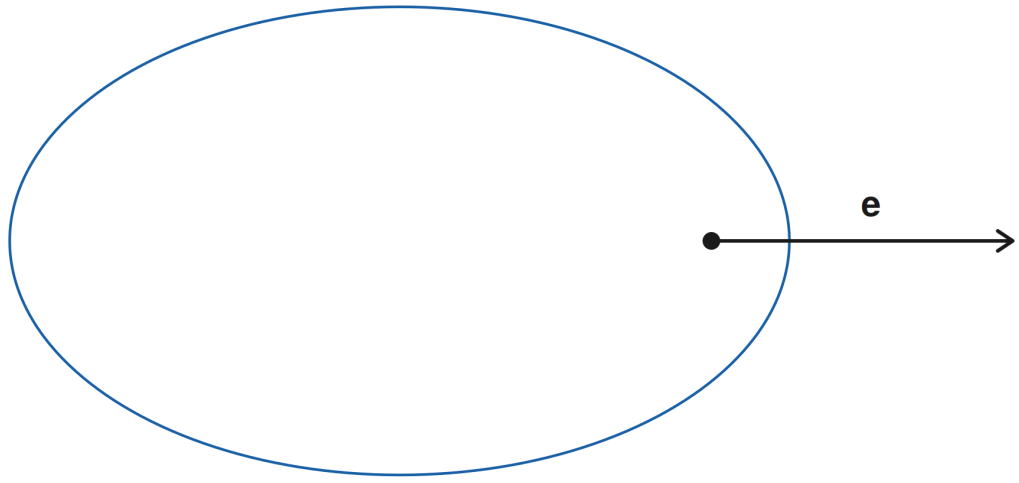
$$E = \tfrac{1}{2}p^2 - \frac{1}{q}$$

and angular momentum

$$\mathbf{L} = \mathbf{q} \times \mathbf{p}$$

conserved, but also another vector:

$$\mathbf{e} = \mathbf{p} \times \mathbf{L} - \frac{\mathbf{q}}{q}$$



$$\mathbf{e} = \mathbf{p} \times \mathbf{L} - \frac{\mathbf{q}}{q}$$

This vector always points in the direction of the orbit's perihelion, and its magnitude e equals the eccentricity of the orbit.

$$\mathbf{e} = \mathbf{p} \times \mathbf{L} - \frac{\mathbf{q}}{q}$$

This extra conserved quantity was named the 'Runge–Lenz vector' after Lenz used it in 1924 to study the hydrogen atom, citing Runge's popular textbook from five years earlier.

But Runge never claimed any originality: he attributed this vector to Gibbs. Now some call it the 'Laplace–Runge–Lenz vector', honoring Laplace's discussion of it in 1799.

But in fact this vector goes back at least to Jakob Hermann, who wrote about it in 1710, triggering further work by Johann Bernoulli in the same year.

I will call $\mathbf{e}$ the **eccentricity vector**.

Which extra symmetries give conservation of the eccentricity vector? Starting from $\{q_j, p_k\} = \delta_{jk}$, some calculations give

$$\begin{aligned} \{L_j, L_k\} &= \epsilon_{jk\ell} L_\ell \\ \{e_j, L_k\} &= \epsilon_{jk\ell} e_\ell \\ \{e_j, e_k\} &= -2E \epsilon_{jk\ell} L_\ell \end{aligned}$$

and of course

$$\{E, L_j\} = \{E, e_j\} = 0$$

since $\mathbf{L}$ and $\mathbf{e}$ are conserved. The factor of $-2E$ above is annoying, but on the region of phase space where $E < 0$ — that is, the space of bound states — we can define a vector

$$\mathbf{M} = \frac{\mathbf{e}}{\sqrt{-2E}}$$

and obtain

$$\begin{aligned} \{L_j, L_k\} &= \epsilon_{jk\ell} L_\ell \\ \{L_j, M_k\} &= \epsilon_{jk\ell} M_\ell \\ \{M_j, M_k\} &= \epsilon_{jk\ell} M_\ell \end{aligned}$$

This gives a Lie algebra isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, as becomes clear if we set

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{L} + \mathbf{M}), \qquad \mathbf{B} = \tfrac{1}{2}(\mathbf{L} - \mathbf{M})$$

and check that

$$\begin{aligned} \{A_j, A_k\} &= \epsilon_{jk\ell} A_\ell \\ \{B_j, B_k\} &= \epsilon_{jk\ell} B_\ell \\ \{A_j, B_k\} &= 0 \end{aligned}$$

But $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{so}(4)$, so conservation of the eccentricity vector must come from a hidden $\mathfrak{so}(4)$ symmetry.

And indeed, the group SO(4) acts on the bound states of the Kepler problem in a way that commutes with time evolution! We can understand this geometrically.

In 1847 Hamilton discovered another remarkable feature of the inverse square force law: the momentum $\mathbf{p}$ moves in a circle!

Solving

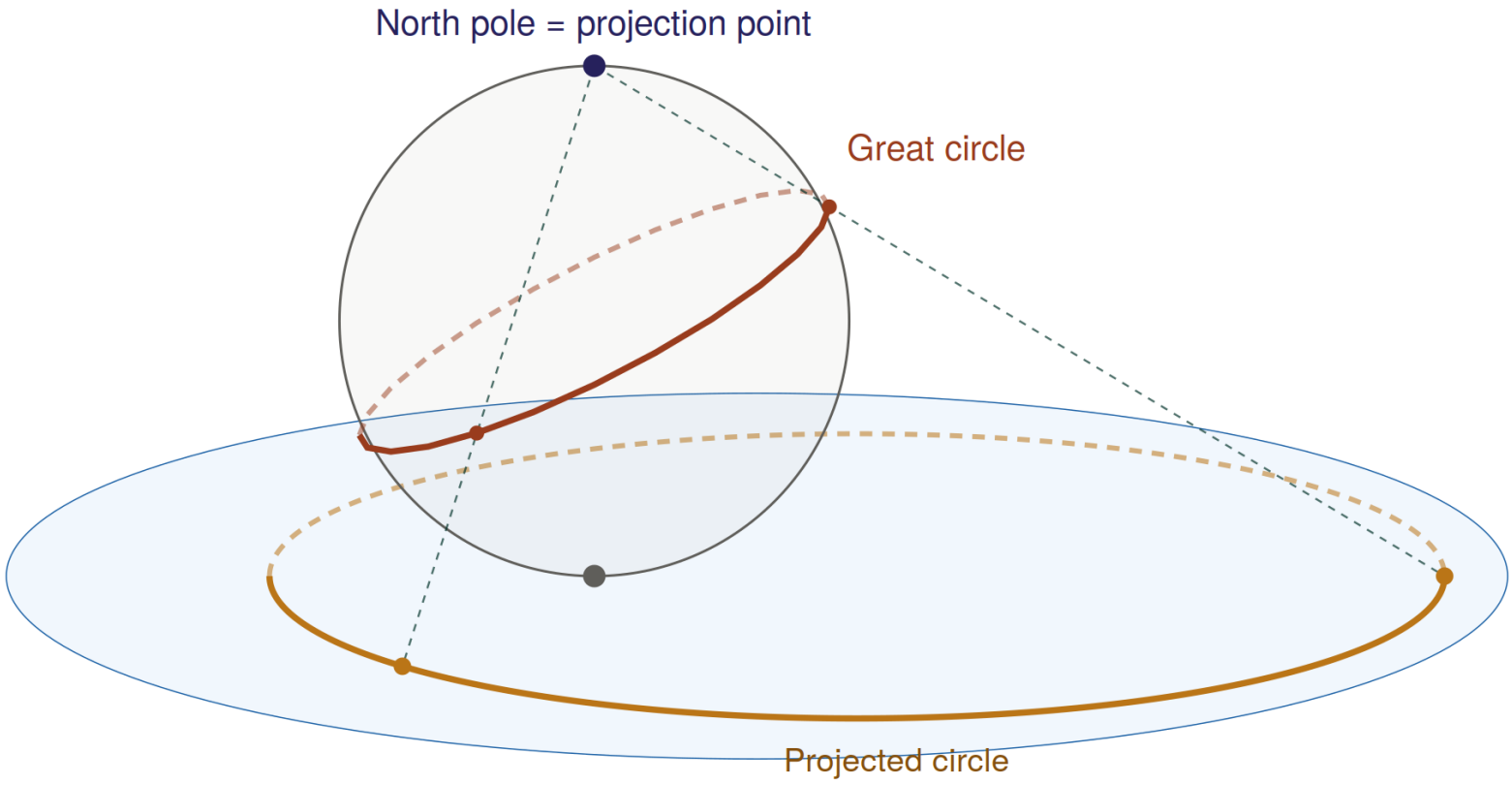
$$\mathbf{e} = \mathbf{p} \times \mathbf{L} - \frac{\mathbf{q}}{q}$$

for $\mathbf{q}/q$, taking the dot product of this with itself, which is 1, and doing some further manipulations, we can show

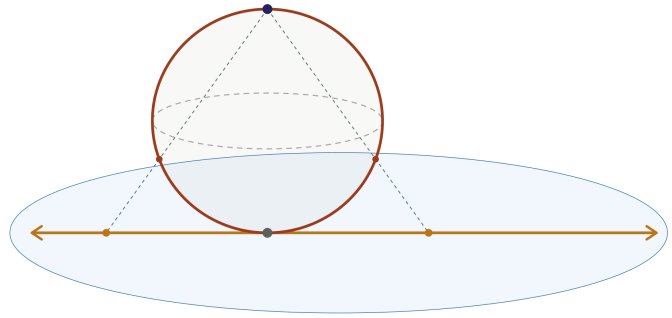
$$\left(\mathbf{p} - \frac{\mathbf{L} \times \mathbf{e}}{L^2}\right)^2 = \frac{1}{L^2}.$$

Thus, $\mathbf{p}$ moves along a circle centered at the point $(\mathbf{L} \times \mathbf{e})/L^2$.

Hamilton's momentum circles are not arbitrary circles in $\mathbb{R}^3$. Using the inverse of stereographic projection, we can map $\mathbb{R}^3$ to the unit 3-sphere. This sends Hamilton's circles in $\mathbb{R}^3$ to great circles in S^3.



Hamilton's construction gives all the great circles in S^3 except those that go through the north and south poles of S^3.



These missing great circles correspond to periodic orbits of infinite eccentricity where a particle starts with momentum zero, falls straight to the origin, then bounces back the way it came!

We can embed the phase space of bound states of the Kepler problem, as a symplectic manifold, into T^*S^3 in such a way that these additional orbits become legitimate trajectories in T^*S^3.

Now every great circle on S^3 corresponds to a periodic orbit of a particle in the attractive inverse square force.

As time passes, the momentum moves along a great circle in S^3. How is the dynamics related to geodesic motion on the 3-sphere?

We can understand this as follows. If E is the Hamiltonian for the Kepler problem — the particle in the attractive inverse square force — a calculation shows

$$-\frac{1}{E} = L^2 + M^2$$

Using the fact that $\mathbf{L} \cdot \mathbf{M} = 0$ and the definitions

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{L} + \mathbf{M}), \qquad \mathbf{B} = \tfrac{1}{2}(\mathbf{L} - \mathbf{M})$$

an easy calculation gives

$$-\frac{1}{E} = 8A^2 = 8B^2$$

In the 3-sphere picture, the observables A_j become functions on T^*S^3. These functions are just the components of momentum for a particle on S^3, defined using a standard basis of right-invariant vector fields on $S^3 \cong \mathrm{SU}(2)$.

Similarly, the observables B_j are the components of momentum using a standard basis of left-invariant vector fields. Since nonrelativistic kinetic energy $\propto$ momentum^2,

$$K = 8A^2 = 8B^2$$

is the Hamiltonian for a nonrelativistic free particle on S^3 with an appropriately chosen mass.

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is the Hamiltonian for a nonrelativistic free particle on S^3 with an appropriately chosen mass. This means

$$K = 8A^2 = 8B^2 = -\frac{1}{E}$$

where E is the energy of the particle in the Kepler problem!

A nonrelativistic free particle on S^3 moves around a great circle at constant speed. Since the Kepler Hamiltonian E is $-1/K$, particles governed by *this* Hamiltonian move along the same trajectories — but typically not at constant speed.

Both K and the Kepler Hamiltonian $E = -1/K$ are well-defined smooth functions on the symplectic manifold that Souriau dubbed the **Kepler manifold**:

$$T^+S^3 = \{(x, p) : \ x \in S^3, p \in T_xS^3, p \neq 0\}.$$

We can also think of

$$T^+S^3 = \{(x, p) : \ x \in S^3, p \in T_xS^3, p \neq 0\}.$$

as the space of light rays in the **Einstein universe**: the manifold $\mathbb{R} \times S^3$ with Lorentzian metric $dt^2 - ds^2$, where ds^2 is the usual metric on the unit 3-sphere. Here a **light ray** is a null geodesic equipped with a choice of covariantly constant tangent vector field, its '4-velocity'.

To describe a light ray using a point in T^+S^3, let $x \in S^3$ be the light ray's position at time zero, while the null cotangent vector $p + \|p\|dt$ describes the light ray's energy-momentum at time zero.

In this way T^+S^3 serves as a phase space for a classical free *massless* relativistic particle in the Einstein universe. The Hamiltonian for such a particle is $\sqrt{K}$.

Summary so far

On the phase space T^*S^3 let

$$K = 8A^2 = 8B^2 \propto \text{momentum}^2$$

We have the following classical Hamiltonians:

- massive nonrelativistic particle in S^3: K
- massless relativistic particle in $\mathbb{R} \times S^3$: $\sqrt{K}$
- Kepler problem: $E = -1/K$.

Quantizing the Kepler problem

We can quantize the bound states of a particle in the inverse square force using the Hilbert space $L^2(S^3)$. A point on S^3 is a momentum eigenstate.

Since $S^3 \cong \mathrm{SU}(2)$, there is a unitary representation R of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $L^2(S^3)$ given by left and right translations.

$$(R(g, g')\psi)(h) = \psi(g^{-1} h\, g')$$

If we call the self-adjoint generators of these left and right translations A_k and B_k $(k = 1, 2, 3)$, we have

$$\begin{aligned} [A_j, A_k] &= i\epsilon_{jk\ell} A_\ell \\ [B_j, B_k] &= i\epsilon_{jk\ell} B_\ell \\ [A_j, B_k] &= 0 \end{aligned}$$

As a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we have

$$L^2(S^3) \cong \bigoplus_j V_j \otimes V_j$$

where $j = 0, \frac{1}{2}, 1, \ldots$ and V_j is the spin-j representation of $\mathrm{SU}(2)$.

The operators A_k act on the first copy of V_j as the usual angular momentum operators on the spin-j representation, while the operators B_k act on the second copy.

Thus, the operator $A^2 = B^2$ has eigenvalue $j(j+1)$ on $V_j \otimes V_j$, and the multiplicity of this eigenvalue is $(2j+1)^2$.

This matches what we see in the hydrogen atom: not counting spin, there are n^2 eigenstates in the nth energy level.

The Hamiltonian for bound states of the *quantum* particle in the inverse square force is

$$H_0 = -\frac{1}{8(A^2 + \frac{1}{4})} = -\frac{1}{8(B^2 + \frac{1}{4})}$$

This differs from the classical Hamiltonian by the extra $+\frac{1}{4}$, which can be explained using the Duflo isomorphism. With this correction, the eigenvalues match the energy levels of the bound states of hydrogen — treated nonrelativistically, and ignoring the electron's spin.

There's a nice geometrical meaning: $A^2 = B^2$ is proportional to the positive definite Laplacian on S^3:

$$A^2 = B^2 = \tfrac{1}{4}\Delta$$

Introducing spin

Now let's introduce spin, but still for bound states of the *nonrelativistic* hydrogen atom. This just doubles the Hilbert space to $L^2(S^3) \otimes \mathbb{C}^2$. The Hamiltonian is now

$$H = H_0 \otimes 1$$

where H_0 is the Hamiltonian we just saw in the spin-zero case.

If we trivialize the spinor bundle on S^3 using the right translation action of SU(2), we can think of $\psi \in L^2(S^3) \otimes \mathbb{C}^2$ as a spinor field on S^3. Then we can show

$$H = -\frac{1}{2(\not{\partial} - \frac{1}{2})^2}$$

where $\not{\partial}$ is the Dirac operator on the 3-sphere.

Chiral massless spin-1/2 particles on $\mathbb{R} \times S^3$ are described by fields

$$\psi \colon \mathbb{R} \times S^3 \to \mathbb{C}^2$$

The **left-handed Weyl equation** is

$$\frac{\partial \psi}{\partial t} = -i \not{\partial} \psi$$

where $\not{\partial}$ is the Dirac operator on S^3. We can use $L^2(S^3) \otimes \mathbb{C}^2$ as a Hilbert space of initial data; then $\not{\partial}$ is an unbounded self-adjoint operator that we can use as the Hamiltonian.

$\mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on this Hilbert space with generators A_k and $B_k + S_k$, S_k being spin angular momentum. $\not{\partial}$ commutes with this group action.

But there's a problem: the Dirac operator $\not{\partial}$ has a spectrum that's unbounded below! To fix this in one-particle relativistic quantum mechanics we can just change the complex structure on $L^2(S^3) \otimes \mathbb{C}^2$ to

$$j = i \frac{\not{\partial}}{|\not{\partial}|}$$

and use the Hamiltonian $|\not{\partial}|$, which has only positive eigenvalues. Notice $i\not{\partial} = j|\not{\partial}|$, so the left-handed Weyl equation

$$\frac{\partial \psi}{\partial t} = -i \not{\partial} \psi$$

is equivalent to

$$\frac{\partial \psi}{\partial t} = -j |\not{\partial}| \psi$$

But now the Hamiltonian is positive! Call $L^2(S^3) \otimes \mathbb{C}^2$ with this new complex structure $\mathbf{H}$.

In fact there's a unitary equivalence

$$F\colon L^2(S^3) \otimes \mathbb{C}^2 \xrightarrow{\sim} \mathbf{H}$$

that commutes with the representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the Dirac operator on each space!

This lets us make both the Hamiltonian for the hydrogen atom with electron spin

$$H = -\frac{1}{2(\not{\partial} - \frac{1}{2})^2}$$

and the Hamiltonian for the left-handed massless spin-1/2 particle

$$|\not{\partial}|$$

into operators on the *same* space. The second operator is a function of the first.

Second quantization

Define a fermionic Fock space by forming the exterior algebra

$$\Lambda(L^2(S^3) \otimes \mathbb{C}^2) \cong \Lambda\mathbf{H}$$

and taking its Hilbert space completion $\mathbf{K}$. States in $\mathbf{K}$ describe collections of fermions, like $\psi_1 \wedge \cdots \wedge \psi_N$ where $\psi_i \in \mathbf{H}$.

On the on hand, $\mathbf{K}$ is the Hilbert space for *noninteracting collections of electrons orbiting a nucleus*: a very simple toy model of an atom.

On the other hand, it's the Hilbert space for *noninteracting collections of left-handed massless spin-*1/2 *fermions in* $\mathbb{R} \times S^3$: a massless spin-1/2 free quantum field.

Any 1-particle Hamiltonian A on $\mathbf{H}$ gives a multiparticle Hamiltonian $d\Lambda(A)$ on $\mathbf{K}$. For example

$$d\Lambda(A)(\psi_1 \wedge \psi_2 \wedge \psi_3) =$$

$$A\psi_1 \wedge \psi_2 \wedge \psi_3 + \psi_1 \wedge A\psi_2 \wedge \psi_3 + \psi_1 \wedge \psi_2 \wedge A\psi_3$$

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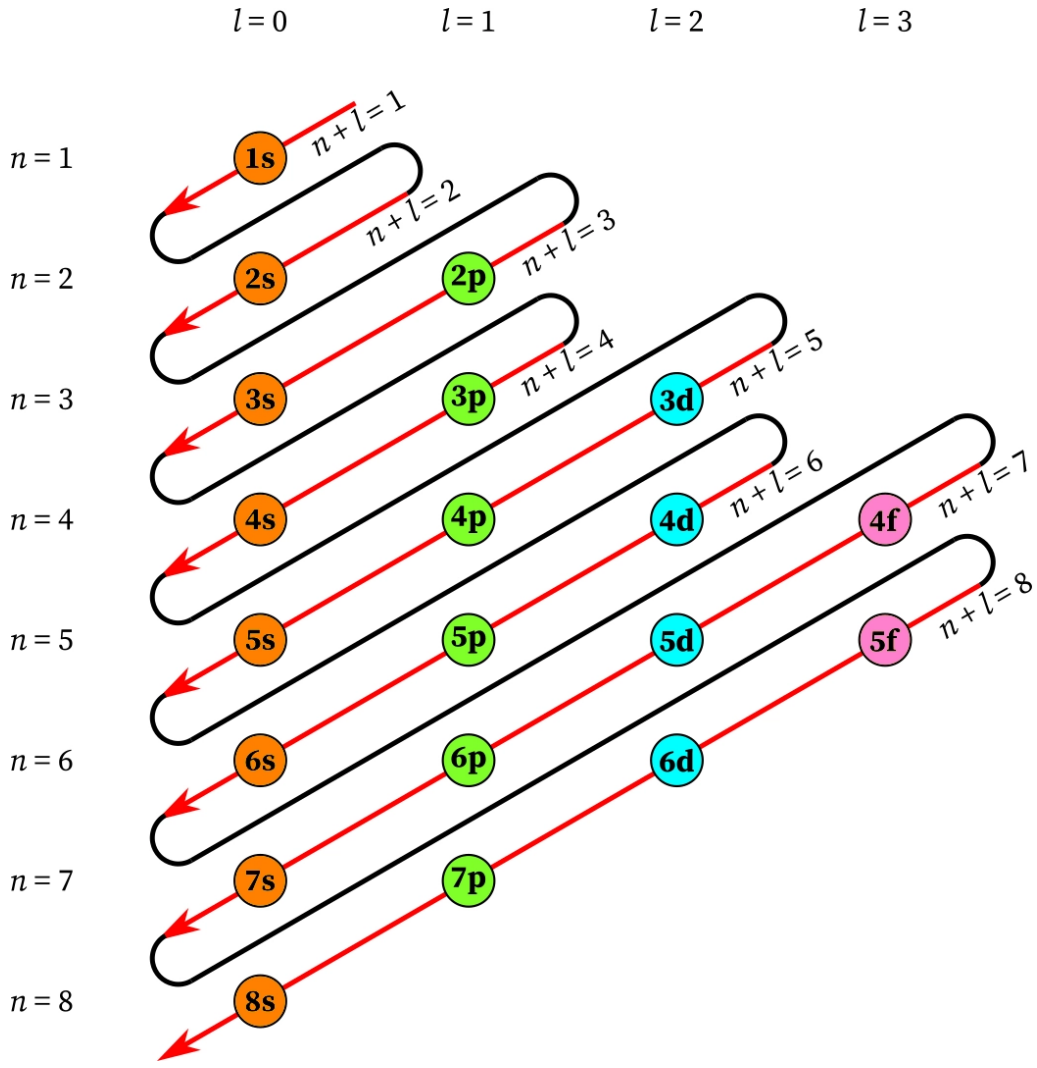
If we start with the Hamiltonian H for the hydrogen atom, and for each N take the lowest-energy N-electron state for the Hamiltonian $d\Lambda(H)$, we don't get the periodic table. Instead, we fill in states of

$$\mathbf{H} \cong L^2(S^3) \otimes \mathbb{C}^2 \cong \bigoplus_j V_j \otimes V_j \otimes \mathbb{C}^2$$

according to increasing values of $j = 0, \frac{1}{2}, 1, \ldots$, with $2n^2 = 2(2j+1)^2$ electrons in each "shell", all with the same energy.

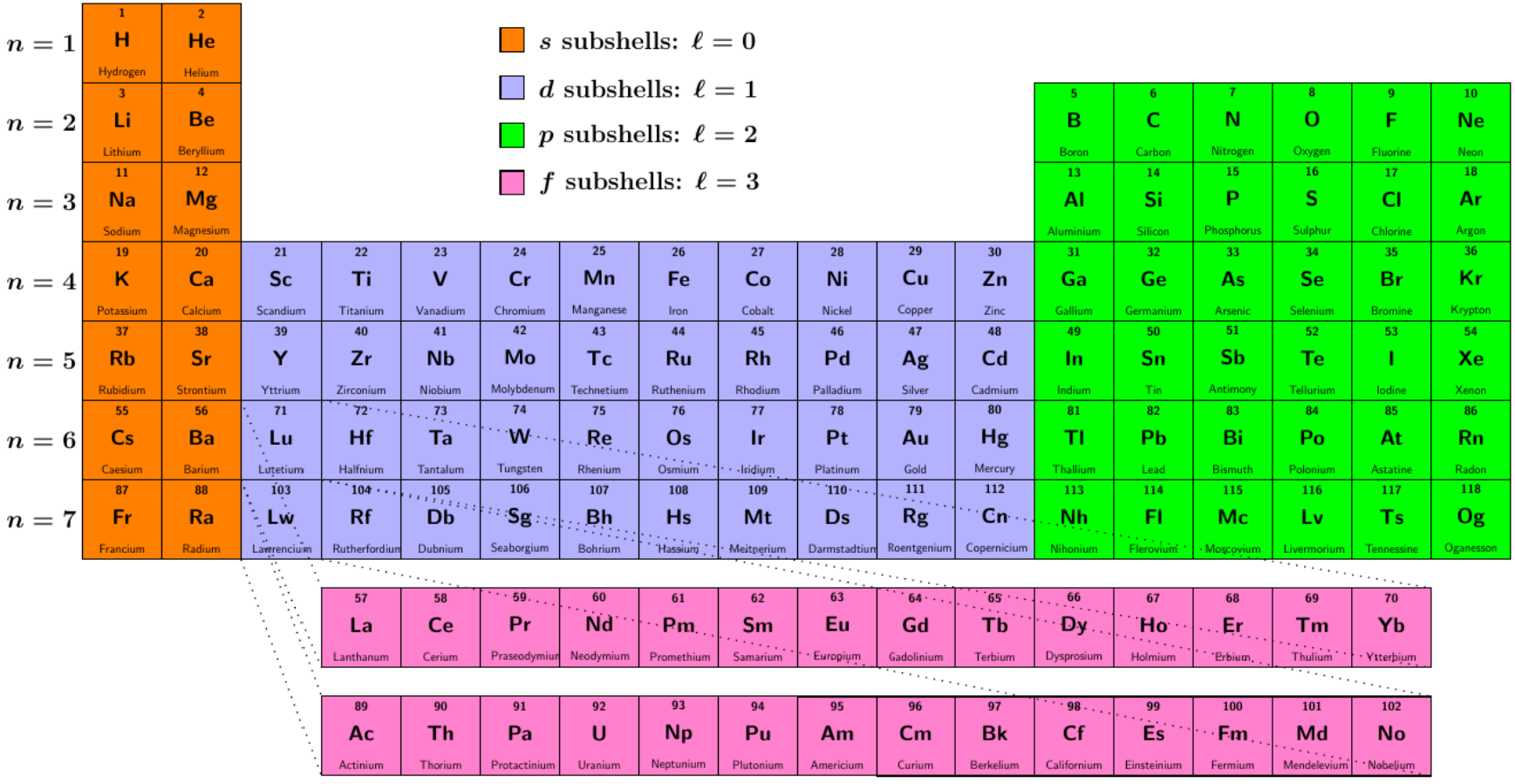
In 1936, Madelung argued for these rules:

- states are filled in order of increasing value of $n + \ell$;
- states with the same value of $n + \ell$ are filled in order of decreasing ℓ.



We can get the Madelung rules if we take a suitable single-particle Hamiltonian H_{single} on $\mathbf{H}$, form the multiparticle Hamiltonian $d\Lambda(H_{\text{single}})$ on $\mathbf{K}$, and for each N take the lowest-energy N-electron state for the Hamiltonian $d\Lambda(H_{\text{single}})$.

So, we get a pretty good approximation to the periodic table from a suitable fermionic quantum field theory on $\mathbb{R} \times S^3$!


$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$
$n = 6$
$n = 7$
s subshells: $\ell = 0$
d subshells: $\ell = 1$
p subshells: $\ell = 2$
f subshells: $\ell = 3$
1 H Hydrogen
2 He Helium
3 Li Lithium
4 Be Beryllium
5 B Boron
6 C Carbon
7 N Nitrogen
8 O Oxygen
9 F Fluorine
10 Ne Neon
11 Na Sodium
12 Mg Magnesium
13 Al Aluminium
14 Si Silicon
15 P Phosphorus
16 S Sulphur
17 Cl Chlorine
18 Ar Argon
19 K Potassium
20 Ca Calcium
21 Sc Scandium
22 Ti Titanium
23 V Vanadium
24 Cr Chromium
25 Mn Manganese
26 Fe Iron
27 Co Cobalt
28 Ni Nickel
29 Cu Copper
30 Zn Zinc
31 Ga Gallium
32 Ge Germanium
33 As Arsenic
34 Se Selenium
35 Br Bromine
36 Kr Krypton
37 Rb Rubidium
38 Sr Strontium
39 Y Yttrium
40 Zr Zirconium
41 Nb Niobium
42 Mo Molybdenum
43 Tc Technetium
44 Ru Ruthenium
45 Rh Rhodium
46 Pd Palladium
47 Ag Silver
48 Cd Cadmium
49 In Indium
50 Sn Tin
51 Sb Antimony
52 Te Tellurium
53 I Iodine
54 Xe Xenon
55 Cs Caesium
56 Ba Barium
71 Lu Lutetium
72 Hf Halfnium
73 Ta Tantalum
74 W Tungsten
75 Re Rhenium
76 Os Osmium
77 Ir Iridium
78 Pt Platinum
79 Au Gold
80 Hg Mercury
81 Tl Thallium
82 Pb Lead
83 Bi Bismuth
84 Po Polonium
85 At Astatine
86 Rn Radon
87 Fr Francium
88 Ra Radium
103 Lw Lawrencium
104 Rf Rutherfordium
105 Db Dubnium
106 Sg Seaborgium
107 Bh Bohrium
108 Hs Hassium
109 Mt Meitnerium
110 Ds Darmstadtium
111 Rg Roentgenium
112 Cn Copernicium
113 Nh Nihonium
114 Fl Flerovium
115 Mc Moscovium
116 Lv Livermorium
117 Ts Tennessine
118 Og Oganesson
57 La Lanthanum
58 Ce Cerium
59 Pr Praseodymium
60 Nd Neodymium
61 Pm Promethium
62 Sm Samarium
63 Eu Europium
64 Gd Gadolinium
65 Tb Terbium
66 Dy Dysprosium
67 Ho Holmium
68 Er Erbium
69 Tm Thulium
70 Yb Ytterbium
89 Ac Actinium
90 Th Thorium
91 Pa Protactinium
92 U Uranium
93 Np Neptunium
94 Pu Plutonium
95 Am Americium
96 Cm Curium
97 Bk Berkelium
98 Cf Californium
99 Es Einsteinium
100 Fm Fermium
101 Md Mendelevium
102 No Nobelium