Classifying Spaces For Topological 2-Groups

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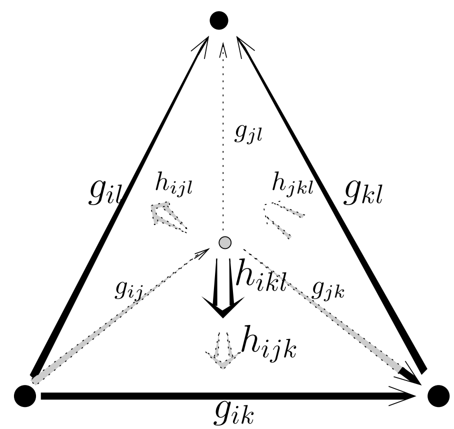


for online references, see:

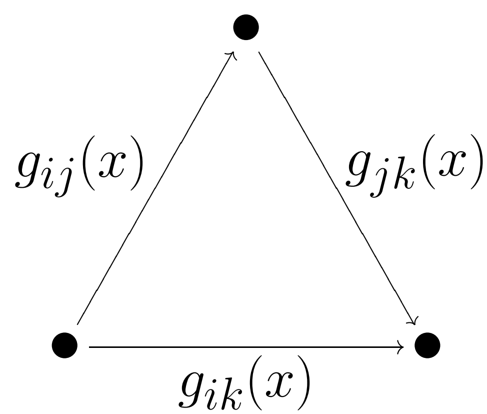
http://math.ucr.edu/home/baez/barcelona/

Čech Cohomology for Bundles

If G is a topological group and M is a topological space, we can describe a principal G-bundle $P \to M$ using a **Čech cocycle**. This consists of an open cover $\mathcal{U} = \{U_i\}$ of M together with **transition functions**

$$g_{ij} \colon U_i \cap U_j \to G$$

such that

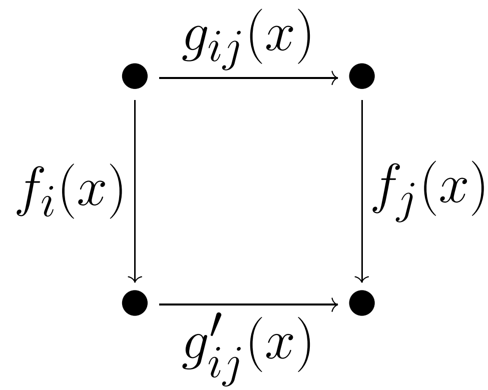


commutes for all $x \in U_i \cap U_j \cap U_k$.

Two Čech cocycles define isomorphic bundles iff they are **cohomologous**, meaning there are functions

$$f_i \colon U_i \to G$$

such that



commutes for all $x \in U_i \cap U_j$.

The set of cohomology classes of Čech cocycles is called $\check{H}(\mathcal{U}, G)$. Taking the inverse limit as we refine the open cover, we obtain the (first) **Čech cohomology** of M with coefficients in G:

$$\check{H}(M, \mathcal{G}) = \varprojlim_{\mathcal{U}} \check{H}(\mathcal{U}, G)$$

There is a bijection between $\check{H}(M, G)$ and the set of isomorphism classes of principal G-bundles over M.

A Famous Old Theorem

Here is the result we'd like to categorify — a result first due to Milnor but polished by Steenrod, Segal, Milgram and May:

Thm. Let G be a well-pointed topological group. Let BG, the **classifying space** of G, be the geometric realization of the nerve of G. Then for any paracompact Hausdorff space M, there is a bijection

$$[M, BG] \cong \check{H}(M, G)$$

(A topological group G is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)

Baas, Bökstedt and Kro have categorified this famous old theorem on classifying spaces, replacing the topological group G by any sufficiently nice *topological 2-category* $\mathcal{C}$. They construct a space $B\mathcal{C}$ such that $[M, B\mathcal{C}]$ classifies '$\mathcal{C}$-2-bundles'.

Here we less ambitiously replace G by any sufficiently nice *topological 2-group* $\mathcal{G}$.

Baas, Bökstedt and Kro classify 2-bundles up to *'concordance'*. We instead describe them using Čech cocycles, and say two cocycles are equivalent when they are *cohomologous*. Our results imply these two equivalence relations are the same (for nice M, $\mathcal{G}$).

Topological 2-Groupoids

Defn. A **2-groupoid** is a strict 2-category where all morphisms and 2-morphisms are strictly invertible.

Defn. A **topological 2-groupoid** $\mathcal{G}$ is a 2-groupoid internal to Top. In other words, it has:

- a topological space of objects,
- a topological space of morphisms,
- a topological space of 2-morphisms,

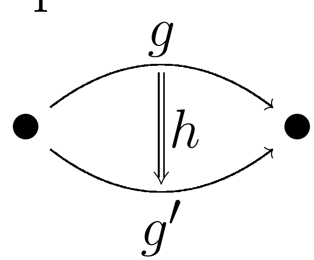
and all the 2-groupoid operations are continuous.

Topological 2-Groups

Defn. A **topological 2-group** is a topological 2-groupoid with one object.

So, it has one object: $\bullet$

together with 1-morphisms: $\bullet \xrightarrow{g} \bullet$

and 2-morphisms: 

This is secretly the same as a 'topological categorical group' or 'topological crossed module'.

The Čech 2-Groupoid

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space M.

Defn. The **Čech 2-groupoid** $\widehat{\mathcal{U}}$ is the topological 2-groupoid where:

- objects are pairs (x, i) with $x \in U_i$,
- there is a single morphism from (x, i) to (x, j) when $x \in U_i \cap U_j$, and none otherwise,
- there are only identity 2-morphisms.

(This is just a topological groupoid promoted to a 2-groupoid by throwing in identity 2-morphisms.)

Čech Cohomology for 2-Bundles

Defn. A **Čech cocycle** with coefficients in a topological 2-group $\mathcal{G}$ is a continuous weak 2-functor $g\colon \widehat{\mathcal{U}} \to \mathcal{G}$.

Defn. Two Čech cocycles g, g' are **cohomologous** if there is a continuous weak natural isomorphism $f\colon g \Rightarrow g'$.

Defn. Let $\check{H}(\mathcal{U}, \mathcal{G})$ be the set of cohomology classes of Čech cocycles. We define the **Čech cohomology** of M with coefficients in $\mathcal{G}$ to be the inverse limit as we refine the cover:

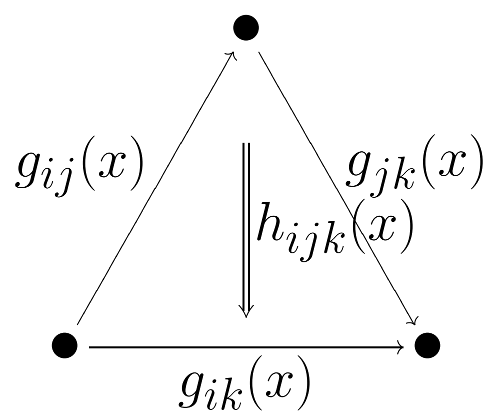
$$\check{H}(M, \mathcal{G}) = \varprojlim_{\mathcal{U}} \check{H}(\mathcal{U}, \mathcal{G})$$

A Čech cocycle $g\colon \hat{\mathcal{U}} \to \mathcal{G}$ is a recipe for building a 'principal $\mathcal{G}$-2-bundle' over M using 'transition functions'.

It sends every object of $\hat{\mathcal{U}}$ to the one object of $\mathcal{G}$, $\bullet$.

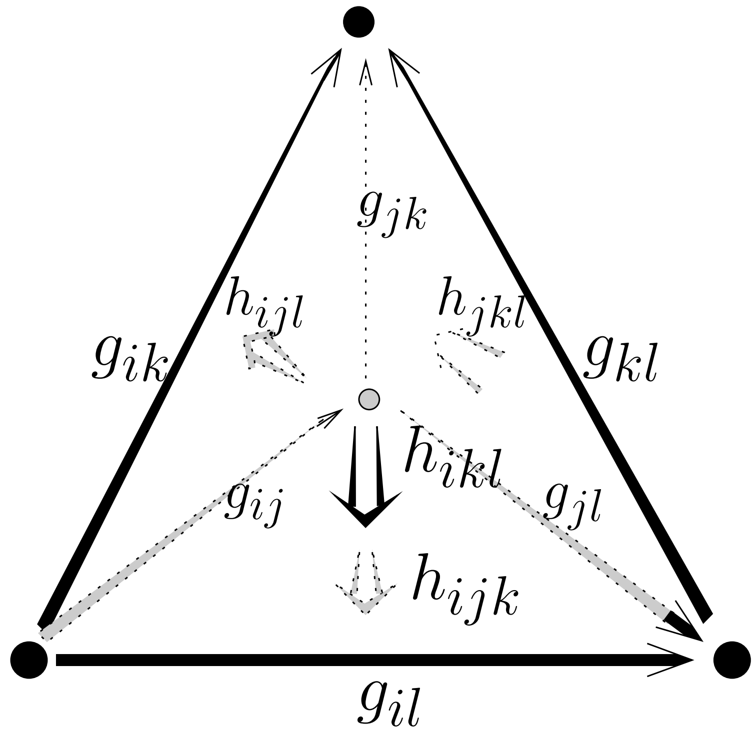
It sends each 1-morphism $(x,i) \to (x,j)$ to a 1-morphism $g_{ij}(x)\colon \bullet \to \bullet$, depending continuously on x.

Composition of 1-morphisms is weakly preserved:



for some 2-morphism $h_{ijk}(x)$ depending continuously on $x \in U_i \cap U_j \cap U_k$.

Finally, the h_{ijk} must make these tetrahedra commute:

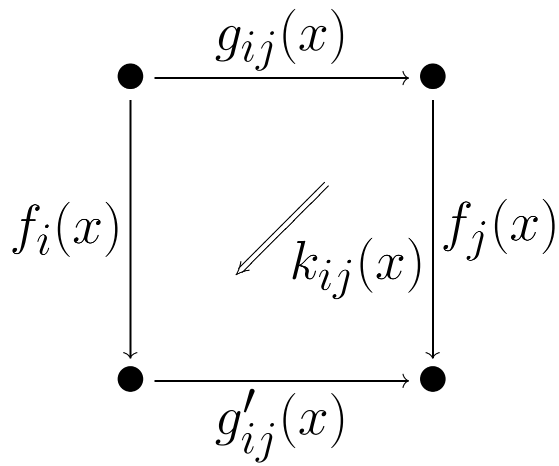


Bartels has shown we can assume without loss of generality that $g_{ii}(x) = 1$ and that $h_{ijk}(x) = 1$ whenever two or more of the indices i, j and k agree. Then we have a **normalized** cocycle.

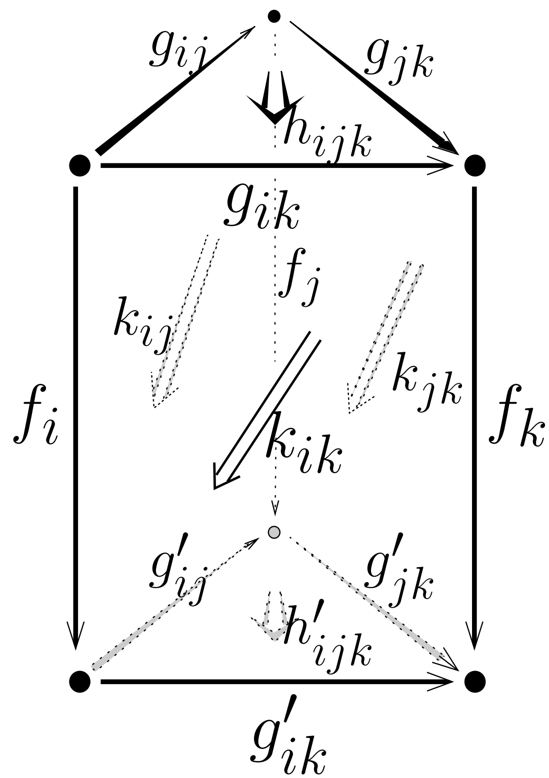
Given Čech cocycles g and g', a continuous weak natural isomorphism $f\colon g \Rightarrow g'$ gives an isomorphism between the corresponding 2-bundles.

f sends each object (x,i) of $\hat{\mathcal{U}}$ to a 1-morphism $f_i(x)\colon \bullet \to \bullet$, depending continuously on x.

It sends each 1-morphism $(x,i) \to (x,j)$ of $\hat{\mathcal{U}}$ to a 2-morphism $k_{ij}(x)$ filling in this naturality square:



Finally, the k_{ij} must make these prisms commute:



Categorifying That Famous Old Theorem

Thm. Suppose $\mathcal{G}$ is a well-pointed topological 2-group and M is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\check{H}(M, \mathcal{G}) \cong [M, B|N\mathcal{G}|]$$

where the topological group $|N\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $B|N\mathcal{G}|$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group G is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of open sets in the cover is contractible.)

The Fine Print

Note: first we think of $\mathcal{G}$ as a group in TopGpd and take its nerve

$$N\colon \mathrm{TopGpd} \to \mathrm{Top}^{\Delta^{\mathrm{op}}}$$

to get a group in simplicial spaces, $N\mathcal{G}$. Then we use geometric realization

$$|\cdot|\colon \mathrm{Top}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$$

to get the topological group $|N\mathcal{G}|$.

Then we think of $|N\mathcal{G}|$ as a 1-object topological groupoid, and take the nerve and the geometric realization *again*, to get the space $B|N\mathcal{G}|$.

A Corollary:
Bundles vs. 2-Bundles

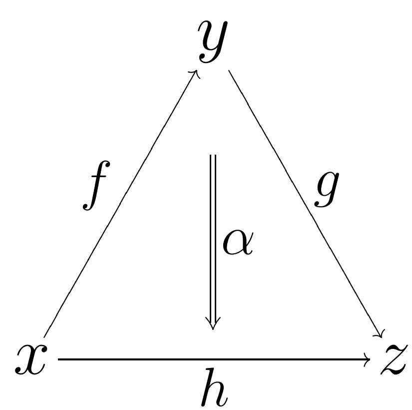
Cor. There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over M
- elements of the Čech cohomology $\check{H}(M, \mathcal{G})$
- homotopy classes of maps $f\colon X \to B|N\mathcal{G}|$
- elements of the Čech cohomology $\check{H}(M, |N\mathcal{G}|)$
- isomorphism classes of principal $|N\mathcal{G}|$-bundles over M.

How We Didn't Prove Our Theorem

As shown by Duskin, every 2-groupoid X has a **nerve**, a simplicial set $\mathcal{N}X$ where:

- 0-simplices of $\mathcal{N}X$ are objects x in X,
- 1-simplices are morphisms $x \xrightarrow{f} y$ in X,
- 2-simplices are triangles in X:



- and so on...

Similarly, the nerve $\mathcal{N}X$ of a topological 2-groupoid X is a simplicial space.

So, the Čech 2-groupoid $\widehat{\mathcal{U}}$ and the topological 2-group $\mathcal{G}$ give simplicial spaces $\mathcal{N}\widehat{\mathcal{U}}$ and $\mathcal{N}\mathcal{G}$.

A Čech cocycle is a continuous weak 2-functor $g\colon \widehat{\mathcal{U}} \to \mathcal{G}$. This gives a simplicial map $\mathcal{N}g\colon \mathcal{N}\widehat{\mathcal{U}} \to \mathcal{N}\mathcal{G}$.

Two Čech cocycles g, g' are cohomologous iff there is a continuous weak natural transformation $f\colon g \Rightarrow g'$. This gives a simplicial homotopy $\mathcal{N}f\colon \mathcal{N}g \Rightarrow \mathcal{N}g'$.

So, we get a map

$$\mathcal{N}\colon \check{H}(\widehat{\mathcal{U}},\mathcal{G}) \to [\mathcal{N}\widehat{\mathcal{U}},\mathcal{N}\mathcal{G}]$$

where the right side consists of *simplicial* homotopy classes of *simplicial* maps between *simplicial* spaces. In fact this is a bijection!

Next we can perform geometric realization, which sends these to homotopy classes of maps between spaces. So, we get a map

$$|\cdot|\colon [\mathcal{N}\widehat{\mathcal{U}},\mathcal{N}\mathcal{G}] \to [|\mathcal{N}\widehat{\mathcal{U}}|,|\mathcal{N}\mathcal{G}|]$$

If we knew this were a bijection, we'd be done. Why?

When $\mathcal{U}$ is an open cover of a paracompact Hausdorff space M, Segal showed

$$|\mathcal{N}\widehat{\mathcal{U}}| \simeq M.$$

When $\mathcal{G}$ is a well-pointed topological 2-group,

$$|\mathcal{N}\mathcal{G}| \simeq B|N\mathcal{G}|,$$

as shown by Bullejos and Cegarra in the case where $\mathcal{G}$ is topologically discrete.

Given all this, for $\mathcal{U}$ a good cover we have:

$$\begin{array}{ccccc} \check{H}(M,\mathcal{G}) & \cong & \check{H}(\mathcal{U},\mathcal{G}) & \cong & [\mathcal{N}\widehat{\mathcal{U}}, \mathcal{N}\mathcal{G}] \\ & & & & \Big\downarrow |\cdot| \\ & & & & [|\mathcal{N}\widehat{\mathcal{U}}|, |\mathcal{N}\mathcal{G}|] \cong [M, B|N\mathcal{G}|] \end{array}$$

So, if we knew

$$|\cdot|\colon [\mathcal{N}\widehat{\mathcal{U}},\mathcal{N}\mathcal{G}] \to [|\mathcal{N}\widehat{\mathcal{U}}|,|\mathcal{N}\mathcal{G}|]$$

were a bijection, we'd conclude $\check{H}(M,\mathcal{G}) \cong [M, B|N\mathcal{G}|]$.

Conjecture: If $\widehat{\mathcal{U}}$ is an open cover of a paracompact Hausdorff space M, and $\mathcal{G}$ is a well-pointed topological 2-group, then

$$|\cdot|\colon [\mathcal{N}\widehat{\mathcal{U}},\mathcal{N}\mathcal{G}] \to [|\mathcal{N}\widehat{\mathcal{U}}|,|\mathcal{N}\mathcal{G}|]$$

is a bijection. More generally: whenever $\mathcal{A}$ and $\mathcal{B}$ are 'sufficiently nice' simplicial spaces,

$$|\cdot|\colon [\mathcal{A},\mathcal{B}] \to [|\mathcal{A}|,|\mathcal{B}|]$$

is a bijection.

How We Did Prove Our Theorem

Our actual proof is less conceptual, but it uses some cute facts.

We want to prove

$$\check{H}(M, \mathcal{G}) \cong [M, B|N\mathcal{G}|].$$

The 'famous old theorem' implies

$$\check{H}(M, |N\mathcal{G}|) \cong [M, B|N\mathcal{G}|]$$

so we just need to show

$$\check{H}(M, \mathcal{G}) \cong \check{H}(M, |N\mathcal{G}|).$$

Corresponding to the topological 2-group $\mathcal{G}$ there is a topological crossed module

$$H \longrightarrow G$$

where as usual, G acts on H. Let $EH \longrightarrow BH$ be the universal H-bundle. Segal has described a natural way to make EH into a topological group. Since G acts on H, it acts on EH, so we may define $G \ltimes EH$.

In fact, there is a short exact sequence

$$1 \longrightarrow H \longrightarrow G \ltimes EH \longrightarrow |N\mathcal{G}| \longrightarrow 1$$

But what is the use of this exact sequence:

$$1 \to H \to G \ltimes EH \to |N\mathcal{G}| \to 1 \quad ?$$

Whenever we have a short exact sequence of well-pointed groups where the projection is a fibration:

$$1 \to A \to B \to C \to 1$$

the left portion $A \to B$ gives a topological crossed module, which we can view as a topological 2-group. As noted by Breen, there is then an isomorphism

$$\check{H}(M, A \to B) \cong \check{H}(M, C).$$

So, we have an isomorphism

$$\check{H}(M, H \to G \ltimes EH) \cong \check{H}(M, |N\mathcal{G}|)$$

But what is the use of this isomorphism:

$$\check{H}(M, H \to G \ltimes EH) \cong \check{H}(M, |N\mathcal{G}|) \quad ?$$

The point is that we can also show

$$\check{H}(M, H \to G \ltimes EH) \cong \check{H}(M, H \to G).$$

The right hand side is just $\check{H}(M, \mathcal{G})$, since $\mathcal{G}$ is another name for the crossed module $H \to G$. Putting these facts together, we obtain

$$\check{H}(M, \mathcal{G}) \cong \check{H}(M, |N\mathcal{G}|)$$

as desired!