Letters in Mathematical Physics **13** (1987) 135–136.

Bell's Inequality for C^*-Algebras

J. BAEZ
Department of Mathematics, Yale University, New Haven, CT 06520, U.S.A.

(Received: 20 October 1986)

Abstract. Bell's inequality dealing with 'local hidden variables' is given two formulations in terms of C^*-algebras. In particular, Bell's inequality holds for all states on $A \otimes B$ whenever A and B are unital C^*-algebras at least one of which is Abelian, i.e., at least one corresponds to a classical physical system.

In this Letter I prove two versions of Bell's Inequality [1] applicable to C^*-algebras. For convenience all C^*-algebras discussed will be assumed to be unital. Let $A \otimes B$ denote the projective tensor product of C^*-algebras A and B. A state ω on $A \otimes B$ is a 'product state' if it is of the form $\omega_1 \otimes \omega_2$ for states ω_1 on A, ω_2 on B. A state ω on $A \otimes B$ is 'decomposable' if it is in the weak-* closure of the convex hull of the product states on $A \otimes B$. Bell's inequality can be viewed as a theorem about decomposable states:

THEOREM 1. *Let A and B be C^*-algebras and let ω be a decomposable state on $A \otimes B$. If $a, a' \in A$ and $b, b' \in B$ are self-adjoint and of norm $\leqslant 1$ then*

$$|\omega(a \otimes (b - b'))| + |\omega(a' \otimes (b + b'))| \leqslant 2.$$

Proof. The proof follows [1]. Suppose $\omega = \omega_1 \otimes \omega_2$. Then

$$\begin{aligned} &\omega(a \otimes (b - b')) \\ &\quad = \omega_1(a)\omega_2(b) - \omega_1(a)\omega_2(b') \\ &\quad = \omega_1(a)\omega_2(b)(1 \pm \omega_1(a')\omega_2(b')) - \\ &\qquad - \omega_1(a)\omega_2(b')(1 \pm \omega_1(a')\omega_2(b)) \end{aligned}$$

so

$$\begin{aligned} &|\omega(a \otimes (b - b'))| \\ &\quad \leqslant |1 \pm \omega_1(a')\omega_2(b')| + |1 \pm \omega_1(a')\omega_2(b)| \\ &\quad \leqslant 1 \pm \omega_1(a')\omega_2(b') + 1 \pm \omega_1(a')\omega_2(b) \\ &\quad \leqslant 2 \pm \omega(a' \otimes (b + b')) \end{aligned}$$

hence,

$$|\omega(a \otimes (b - b'))| + |\omega(a' \otimes (b + b'))| \leqslant 2.$$

If ω is a convex combination of product states, $\omega = \Sigma c_i \omega_i$, the above implies

$$|\omega(a \otimes (b - b'))| + |\omega(a' \otimes (b + b'))|$$
$$\leqslant \Sigma c_i \{ |\omega_i(a \otimes (b - b'))| + |\omega_i(a' \otimes (b + b'))| \} \leqslant 2 .$$

If ω is a weak-* limit of such convex combinations the inequality holds by continuity. □

If A and B are the C^*-algebras corresponding to two physical systems, the product system has C^*-algebra $A \otimes B$, and admits 'local hidden variables' in Bell's sense when all its states are decomposable. This happens if *at least one* of the two systems is classical:

PROPOSITION. *If either A or B is Abelian, all states on $A \otimes B$ are decomposable.*

Proof. As in Theorem IV 4.14 of [2], one can show that every pure state on $A \otimes B$ is a product of pure states. (The theorem deals with the injective tensor product but the proof carries over without modification.) Thus, every state on $A \otimes B$ is decomposable. □

Using this one can obtain another formulation of Bell's inequality:

THEOREM 2. *If A and B are C^*-subalgebras of a C^*-algebra C such that $[A, B] = 0$, and either A or B is Abelian, then*

$$|\omega(a(b - b'))| + |\omega(a'(b + b'))| \leqslant 2$$

for any state ω on C and self-adjoint $a, a' \in A$, $b, b' \in B$ with norm $\leqslant 1$.

Proof. By Proposition IV 4.7 of [2] there is a *-homomorphism $\rho: A \otimes B \to C$ such that $\rho(a \otimes b) = ab$ for all $a \in A$, $b \in B$. Thus, for all $a \in A$, $b \in B$, $\omega(ab) = \omega \circ \rho(a \otimes b)$, where $\omega \circ \rho$ is a state on $A \otimes B$. By the Proposition above, $\omega \circ \rho$ is decomposable, so by Theorem 1 the desired result follows. □

If A and B are non-Abelian type I C^*-algebras, the usual counter-example involving 2×2 matrices [1] shows that Bell's inequality (as formulated in Theorem 1) fails for certain indecomposable states. For applications to quantum field theory it would be interesting to see if the assumption that A and B be type I can be dropped.

References

1. Bell, J. S., in B. d'Espagnat (ed.), *Proceedings of the International School of Physics, 'Enrico Fermi'*, Academic Press, New York, 1971, p. 171.
2. Takesaki, M., *Theory of Operator Algebras* I, Springer-Verlag, New York, 1979.