Braids and Quantization

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1 The Conway Polynomial

These days I'm mainly working on the relationship of braids and quantization. Lots of people are interested in that these days, but lots more aren't, I bet, so let me briefly explain just a bit...

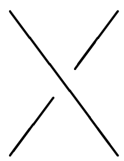
There's a knot invariant called the Conway polynomial that may be defined by essentially two rules. It's a polynomial in one variable, say z; let's call the polynomial assigned to the knot (or link) K, $\nabla(K)$. The knot K has to be oriented; that is, one must draw little arrows tangent to it that say which way to go: $\rightarrow$. Okay:

Rule 1: If K is the unknot (an unknotted circle), $\nabla(K) = 1$. This is sort of a normalization rule.

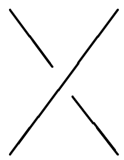
Rule 2: Suppose K, K', and L are 3 knots (or links) differing at just one crossing (we're supposing them to be drawn as pictures in 2 dimensions).

At this crossing they look as follows:

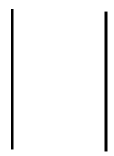
K looks like:



K' looks like:



L looks like:



(All of them should have arrows pointing down. Any rotated version of this picture is fine too — this is topology, after all!)

Then we have the "skein relation"

$$\nabla(K) - \nabla(K') = z\nabla(L).$$

It was Louis Kauffman, I believe, who first noted that this looks a lot like the famous canonical commutation relations, or Heisenberg relations:

$$pq - qp = -i\hbar$$

(Here p is momentum, q is position, and $\hbar$ is Planck's constant). Of course it looks more like it if you call the variable z "$\hbar$", but the real thing is to note that the two kinds of crossings in K and K' are analogous (somehow) to the different orderings in pq and qp.

Well, one could easily laugh this off as the ravings of someone who has been studying knot theory for too long, but it turned out that there *was* a deep connection. It was Turaev who first gave a precise formulation. He constructed an algebra from knots that involved a variable, $\hbar$, and such that as $\hbar \to 0$ it converged (in some sense) to an algebra of loops on a two-dimensional surface. Projected onto a two-dimensional surface the knots K and K' are the same, of course, so this makes some sense.

This however was only the tip of the iceberg...

2 Anyons and Braids

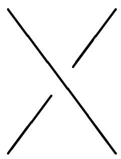
In my previous post I was discussing the Conway polynomial. In the early 80's Vaughn Jones came up with a shockingly similar knot invariant. Interestingly, he arrived at it via von Neumann algebras, a kind of operator algebra invented by von Neumann to study the foundations of quantum physics (among other things). With the Jones polynomial and Conway polynomial around people immediately started searching for a generalization that would encompass both and soon a whole bunch of people found one. It's often called the HOMFLY polynomial these days after the initials of some (but not all) of its discoverers. This "polynomial" is really a Laurent polynomial in two variables, namely a polynomial in x, x^{-1}, and z. It may be calculated by the following rules:

Rule 1: If K is the unknot (an unknotted circle), $G(K) = 1$. This is sort of a normalization rule.

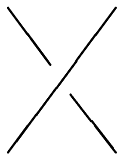
Rule 2: Suppose K, K', and L are 3 knots (or links) differing at just one crossing (we're supposing them to be drawn as pictures in 2 dimensions).

At this crossing they look as follows:

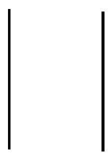
K looks like:



K' looks like:



L looks like:



(Again, all of them should have arrows pointing down, and any rotated version of this picture is fine too.)

Then we have the "skein relation" (here's where it's different from the Conway polynomial):

$$xG(K) - x^{-1}G(K') = zG(L).$$

Nice and simple! (Though it's not at all trivial to prove its existence and uniqueness!) Note that the variable x keeps track of handedness or what a physicist might call "parity" — it's an easy exercise to show that if the knot K^* is the mirror image of K then

$$G(K^*)(x,z) = G(K)(x^{-1},z)$$

It's a pleasant exercise to use rules 1) and 2) to calculate this polynomial for the left-handed and right-handed trefoil knot and see that the answers are different.

Now the question becomes, what is this polynomial trying to tell us? Here I need a digression into the theory of braids, and, to keep the physicists interested, I'll begin with anyons.

We all know that particles come in two fundamentally different flavors: bosons and fermions. The argument for this is simple, well-known, and a bit misleading. It goes like this. Say we have a bunch of n identical particles. Their state is described by a wave vector (a vector in a complex Hilbert space). We may permute the particles without really doing anything since they are identical, you can think of it as just permuting their "labels". Now in quantum mechanics two wave vectors which differ only by a phase (a scalar factor of unit modulus) describe the same physics. Thus we must have a representation of the symmetric group S_n on the Hilbert space, and it must map any permutation to a scalar multiple of the identity. There are only 2 such representations (another charming exercise): the trivial representation and the one mapping each permutation to its sign. In the former case we say the particles are bosons, and in the latter, fermions.

This seems to be the case in reality. Interestingly, all the fundamental particles one might call "matter" are fermions (quarks and leptons), while all the particles one might call "force fields" are bosons (the photon, W, Z, and gluons). Here of course I am skirting the issue of the Higgs particle, that curious fudge factor. If the Japanese decide to pay for the superconducting supercollider we will see if the Higgs exists.

If one pays close attention to the argument, however, it's full of holes. First of all, why do we really need a representation of S_n — in quantum mechanics a projective representation is good enough! Secondly, if one considers representations where S_n does not act as scalars but as an "internal symmetry group" one gets even more possibilities. These were investigated under the name of parastatistics. Anyway, one can come up with a better argument, the spin-statistics theorem, in relativistic quantum field theory, and that, together with the fact that parastatistics can be redescribed as fermions and bosons in disguise, seems to give a solid explanation for why all we see is bosons and fermions. (Though I couldn't say I'm very familiar *myself* with the whole story.)

Now for the catch: the spin-statistics theorem only holds for spacetimes of dimension 4 and up. You could just say "thank heavens! that just happens to apply to *our* universe!" and leave it at that, or you could note that it's occaisionally possible to *simulate* universes of lower dimension. Take, for example, a thin 2-dimensional layer of stuff: this can act like a little 3-dimensional spacetime. Similarly, filaments can act like 2-dimensional universes. These days condensed matter theorists delight in the odd processes that occur in these contexts, and it was only a matter of time before someone noted that one can, at least in principle, arrange to get particles that are neither bosons or fermions. Wilczek is generally credited with taking the idea of these "anyons" seriously, though it had occured to others earlier.

Here's how it goes in its most primitive form. Say we have some tubes of magnetic flux moving around. (One can play with these flux tubes using superconductors, for example). As long as these tubes stay parallel the problem is essentially a 2-dimensional one: pick a plane perpendicular to the flux tubes and just pay attention to their intersection with that plane. Each tube intersects the plane in a spot which we will regard as a "particle". In each spot there is a **B**-field perpendicular to the plane, and going around each spot is an **A**-field whose curl is the **B**-field. If you like you can think of each spot as a "vortex" of the **A**-field. Now suppose — and here I don't know if this has ever been experimentally achieved — that each tube is electrically charged. In our planar picture then, we've got these "particles" which are charged, each also being a vortex of the **A**-field. Let's assume that all these particles are identical. Now let's see what happens when we interchange two of them. Recall that when you move a particle with charge e through an **A**-field, its phase changes by

$$e^{i\theta}$$

where θ is the line integral of the **A**-field along the path traversed by the particle. Thus when we interchange two of our particles — and here I mean you physically "grab" them and move them around each other so that they trade places! — the wave vector of the system is changed by a phase. I'll let you calculate it. The point is, depending on the charge and the strength of the magnetic flux tube, one can arrange for this phase to be whatever one likes — any complex number of magnitude one! If this number equals 1 one has bosons, if it's -1 one has fermions, but otherwise one has *anyons*.

One can have fun playing with this idea. Many people have. Wilczek is a big proponent of an anyonic theory of high-temperature superconductivity, although recent experiments, demonstrating an apparent absense of spontaneous parity violation that one would expect in this theory, seem to rule it out. It's not 100% dead, though, and in my opinion it's so beautiful that someone should try to make a superconductor based on this principle just for the glory of it. Something that anyone who has followed me so far can have fun doing, is to see what sort of particle a bound state of anyons acts like. It's well-known that two fermions together act like a boson, two bosons act like a boson, and a boson and a fermion act like a fermion . . . extend this to anyons.

A question for the real physicists out there: has anybody ever *really made* anyons and played with them yet?

Now anyons have a lot to do with braids because, as you may have noticed, I have covertly stopped thinking of the the operation of interchanging identical particles as an abstract "switching" — modelled by the symmetric group — and started thinking of it as moving one particle around another. If one draws the worldlines of some anyons as one moves them around each other this way, one has — a braid! I.e., out with the symmetric group, in with the braid group!

The high-handed manner in which I've thrown out the symmetric group and started working with braid group statistics *should* disturb you, but again I can cite fancy mathematical physics papers which should allay your fears. A very nice one is "Local Quantum Theory and Braid Group Statistics" by Froehlich and Gabbiani, which gives a proof of the generalized spin-statistics theorem that holds in 2 and 3 dimensions.

So now we see a close relation between quantum theory — to be precise, "statistics" in quantum theory — and the braid group. Looking back at Kauffman's original insight into the relation of knots and quantum mechanics, it's not blindingly obvious what *that* has to do with *this*! Nonetheless it's all part of one story. (A story which I strongly feel is far from over.) More later.

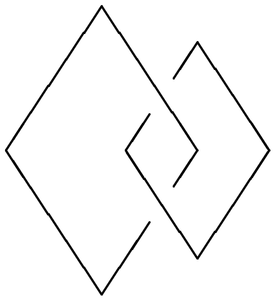
3 Polyakov's Model

Some people have written saying they enjoy these "Braids and quantization" articles, so I'll keep 'em coming. Some also wrote saying that the best known explanation for the mysterious "fractional quantum Hall effect" involves anyons, and referred me to a paper of Frohlich and Zee, "Large Scale Physics of the Quantum Hall Fluid," Nuc Phys B364, 517–540. The Hall effect, recall, is fact that if one applies a magnetic field perpendicular to current in a wire, this makes the electrons wnat to veer off to one side (recall the force is proportional to $\mathbf{v}\times\mathbf{B}$), and they do until the increased charge density on that side creates enough electric field to keep more from crowding over to that side. One observes this by measuring the electric field. The "quantum Hall effect" refers to the charming fact that this effect is quantized for a sufficiently good (cold) conductor: as one increases the magnetic field the resulting electric field goes up like a *step function*; in the right units, the effect takes a jump when the magnetic field is an integer. This effect is reasonably well understood, they say (though I've heard murmurs of discontent occaisionally). I know that Belissard has written nicely about the relationship with noncommutative differential geometry a la Connes. The *fractional* quantum Hall effect refers to the fact that not only at inegers, but also at some *fractions*, one sees a jump. I will enjoy learning how this could result from fractional (i.e., anyonic) spin and statistics.

Now, though, I feel like rambling on a bit about Polyakov's construction of anyons by adding a Hopf term to the Lagrangian of a certain nonlinear sigma model. This is actually used in the anyonic theory of high-Tc superconductivity, but even if that theory is a bunch of baloney, Polyakov's idea is a charming bit of mathematical-physical speculation. It's a nice introduction to solitons, topological quantum field theories, and Witten's explanation of the new knot polynomials in terms of topological quantum field theories.

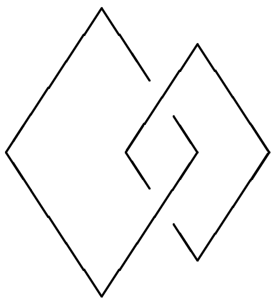
Let's say we have two-dimensional magnet. (Ferro- or antiferro- doesn't really matter at the level of vagueness I'll be working at; the high-Tc superconductors are layered crystals that are antiferromagnets in each layer.) We'll just naively assume each atom has a spin which is a unit vector, i.e. a point on S^2. And we'll just model the state of the magnet as a spin field, that is, a map from space, $\mathbf{R}^2$, to S^2. (I.e. we're doing a continuum limit: for antiferromagnets we flip over the spin of every other atom (in our model) to get a nice continuous map from $\mathbf{R}^2$ to S^2.) Let's assume that all the spins are lined up at spatial infinity. Thus we can add a point at infinity to $\mathbf{R}^2$ (getting a sphere S^2) and describe the state of our magnet as a map from S^2 to S^2. Physicists like to use the delightfully uninformative term "nonlinear sigma model" to describe a field theory in which the field is a map from one manifold (e.g. S^2) to another (e.g. S^2) — a generalization of the usual vector or tensor field. So we've got ourselves a simple nonlinear sigma model to describe the 2d magnet. I should tell you the Lagrangian but I'm carefully avoiding any equations, so I'll just say (for those in the know) that it's the one that gives harmonic maps.

Now, maps from S^2 to S^2 come in various homotopy classes, that is, different maps from S^2 to S^2 may not be able to be continuously deformed into each other. It is a little hard for me to draw these things, but they really aren't hard to visualize with some work. Just as the homotopy classes of maps from S^1 to itself are indexed by an integer, the winding number, so are the maps from S^2 to itself: there's a kind of "winding number" that counts (with sign) how many times you've wrapped the sphere over itself. These twists in the field act sort of like localized particles (for a lower-dimensional analogy imagine them as twists in a ribbon) and are called topological solitons. For physicists, the "winding number" I mentioned above is called the soliton number. It acts like a conserved charge. One can start with a field configuration with zero soliton number – all spins lined straight up – and then have a soliton-antisoliton pair form, move around, and then annihilate, for example. Note that if we track the birth and death of soliton-antisoliton pairs over time by drawing their worldlines, we get a link! This is where knots and braids sneak into the picture:



In this picture time goes up the page, and we see first one pair formed, then another, then they move around each other and then they annihilate. We have a *link* with linking number 1 (let's say — the sign actually depends on a right-hand rule, but since I'm left-handed I object to using the usual right-hand rule).

Polyakov's trick was to add a term to the Lagrangian which equals a constant θ times the linking number. It's a bit more technical so before describing how he gets a local expression for this term I'll just say what its effect is on the physics. Classically, it has no effect whatsoever! Since a small variation in the field configuation doesn't change the linking number (which after all is a topological invariant), the Euler-Lagrange equations (which come from differentiating the Lagrangian) don't notice this term at all. Quantum mechanically, however, one doesn't just look for an extremum of the action. Instead one forms a path integral a la Feynman, integrating $\exp(i\,\mathrm{Action})$ over all histories. So if two histories have different linking numbers, their contribution to the integral will differ by a phase. For example, the configuration above has the exact same action as this one:



by symmetry, except that the first, "right-handed", history has linking number 1, while the second, "left-handed" one has linking number -1. Thus the first will appear in the path integral with a factor of $e^{i\theta}$, while the second will appear with a factor of $e^{-i\theta}$. Thus if one soliton goes around another we get a phase factor, so — here the reader needs a bit of faith — they act like anyons.

Now let me describe Polyakov's term in the Lagrangian in a bit more detail. Here I'll allow myself to be a tad more technical. Let us assume that (as in the pictures above) we are considering histories which begin and end with all spins lined up. Thus our map from spacetime (2d space, 1d time) to S^2 may be regarded as a map from S^3 to S^2, using the old "point at infinity" trick again. The homotopy classes of maps from S^3 to S^2 are also indexed by an integer, this being called the Hopf invariant. The first way to calculate the Hopf invariant shows why Polyakov uses it as an extra term in the Lagrangian. Take a map from S^3 to S^2. By Sard's theorem almost every point in S^2 will have as its inverse image in S^3 a collection of nonintersecting closed curves (i.e., a link). The Hopf invariant may be calculated as follows: take two such points in S^2 and call their inverse images in S^3, L and L'. The Hopf invariant is the linking number $\mathrm{link}(L, L')$ (which doesn't depend on which points you picked). To see how this relates to the story above, take two nearby points in S^2 and draw an arc between them. The inverse image of this arc in S^3 is a "ribbon" or "framed link," and the Hopf invariant, $\mathrm{link}(L, L')$, is also called the "self-linking number" of the framed link, since it includes information about how the ribbon twists, as well as how it links itself when it has more than one connected component. Physically, the contribution to the Hopf invariant due to ribbon twisting is interpreted as due to the rotation of individual anyons. Since spin as well as statistics contributes to the phase $\exp(i\,\mathrm{Action})$, to be precise one must model the anyon trajectories not by a link in spacetime, but by a framed link, which keeps track of how they rotate.

The second way to calculate the Hopf invariant shows how to write it down as an integral over S^3 of a local expression (Lagrangian density). Take the volume form on S^2 and pull it back to S^3 by our map. We now have a closed 2-form on S^3 so we can write it as dA for some 1-form. Now integrate $A \wedge dA$ over S^2 and divide by something like 4π. This is the Hopf invariant! I leave it as an easy exercise to show that it didn't depend on our choice of

A, and as a slightly harder exercise to show that it really is a diffeomorphism invariant, and as a harder exercise to show that this definition of the Hopf invariant agrees with the linking number one. (For more info read Bott and Tu's *Differential Forms and Algebraic Topology.*)

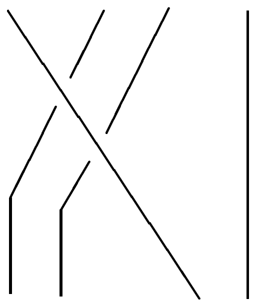
Note that the freedom of choice of A here is none other than what physicists call "gauge freedom." What we have here, in other words, is a gauge theory with a diffeomorphism invariant Lagrangian. (That is, if we keep the Hopf term and drop the harmonic action.) Such theories give boring classical dynamics, because the action is constant on each connected component of the path space. (Or, in physics lingo, the Lagrangian is a total divergence.) But they can give nontrivial dynamics after quantization, because of phase effects. In fact, the simplest example of this sort of deal is the Bohm-Aharonov effect. The particle can go around an obstacle in either of two ways so the path space consists of two components. Classically, a term in the action that is constant on each component doesn't do anything. But quantum-mechanically it leads to interference due to a shift of phase.

These days the ultrasophisticated mathematical physicists and topologists love talking to each other about "topological quantum field theories" in which the Lagrangian is a diffeomorphism invariant. The term with action equal to the integral of $A \wedge dA$ is called the "$U(1)$ Chern-Simons theory", because a 1-form may be regarded as a connection on a $U(1)$ bundle. This is a very simple theory; the more interesting ones use nonabelian gauge groups. Witten showed (in his rough-and-ready manner) that just as the linking number is related to the $U(1)$ Chern-Simons theory, the Jones polynomial is related to the $SU(2)$ Chern-Simons theory. (Many people have been trying to make this more rigorous. Right now my friend Scott Axelrod is working with Singer on the perturbation theory for Chern-Simons theory, which should make the story quite precise.)

4 The Yang-Baxter Equation

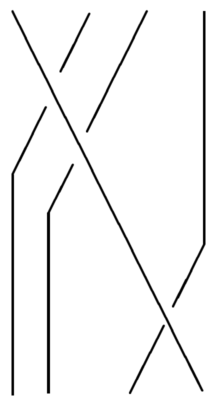
I find that Polyakov model I described last time to be a great example of all sort of things: solitons, instantons, anyons, nonlinear sigma models, gauge theories, and topological quantum field theories all in one! But I want to get back to braids plain and simple and introduce the Yang-Baxter equations. I'll tone down the math for a while so that people who know only a tad of group theory can at least get the definition of the braid group.

So: braids with n strands form a group, called the braid group B_n. (Let's take $n = 4$ as an example.) Multiplication is just defined by gluing one braid onto the bottom of another. For example this braid:

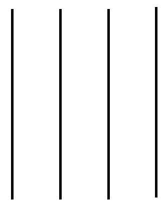


times this one:

equals this:

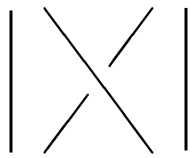


The identity braid is the most boring one:



I leave it as a mild exercise to show that every braid x has an "inverse" y such that $xy = yx = 1$.

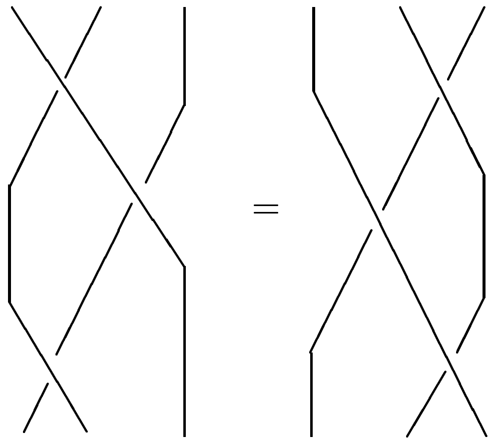
The braid group has special elements $s_1, s_2, \ldots, s_{n-1}$, where s_i is the braid where the i-th strand goes over and to the right of the $(i+1)$-st. For example, s_2 equals



Now, the interesting thing about these "elementary braids" is that s_i and s_j commute if $|i-j| > 1$ (check it!) but s_i and s_{i+1} get tangled up in each other. They satisfy a simple equation, however, the Yang-Baxter equation. (This was known ages before Yang and Baxter came along, and its importance was perhaps discovered by Artin.) Namely,

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

Let me draw it for the braid group on three strands, where $i = 1$:



Draw it for yourself and check that these are topologically equivalent braids (i.e., you can get from one to the other by a little stretching and bending).

Okay, now I'll start turning the math level back up. The braid group is obviously generated by the elementary braids s_i, but the extremely non-obvious fact is that the braid group is isomorphic to the group with the s_i as generators and the relations

$$\begin{aligned} s_i s_j &= s_j s_i && \text{for } |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

Note that there's a homomorphism from the braid group onto the symmetric group with s_i mapped to the ith elementary transposition. In fact the symmetric group may be obtained by throwing in the relations

$$s_i s_i = 1$$

in with the above ones. In other words — and this has a lot of physical/philosophical significance — the braid group may be thought of as the obvious generalization of the group of permutations to a situation in which "switching" two things' places twice does *not* get you back to the original situation. In physics, of course, this happens with anyons.

It's easy to determine the one-dimensional unitary representations of the braid group. In such a representation each s_i gets mapped to a complex number of unit magnitude, which we'll just call s_i. Then the Yang-Baxter equation shows that $s_i = s_{i+1}$, so all the s_i's are equal. The other braid group relations are automatic. Thus for each angle θ there is a unitary rep of the braid group with

$$s_i \to e^{i\theta}$$

and these are all the one-dimensional unitary reps. In physics lingo these are all "fractional statistics." Note that for one of these reps to quotient through the symmetric group we must have $e^{i\theta} = \pm 1$; these correspond to bosonic and fermionic statistics respectively.

Okay, how about reps of the braid group that are not necessarily one-dimensional? Well, suppose V is a vector space and we want to get a rep of B_n on the n-th tensor power of V, which I'll call V^n. Here's a simple way: just map the i-th elementary braid s_i to the linear map taking

$$v_1 \otimes v_2 \otimes ... \otimes v_n$$

to

$$v_1 \otimes \ldots \otimes R(v_i \otimes v_{i+1}) \otimes \ldots \otimes v_n$$

where R is an invertible linear transformation of V^2. In other words, we use R to "switch" the i-th and $(i+1)$-st factors, and leave the rest alone. It's easy to see that this defines a rep of the braid group as long as we can get the Yang-Baxter equation to hold, and this is equivalent to having

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

where I is the identity on V. Actually, *this* equation is more usually known as the Yang-Baxter equation.

Now, Baxter invented this equation when he was looking for problems in 2d statistical mechanics that were exactly solvable (i.e. one could calculate the partition function). A simple example is the "six-vertex" or "ice-type" model, so-called because it describes flat square ice! (Theoretical physicists are never afraid of simplifying the world down to the point where they can understand it.) But there are zillions of examples — take a look at his book on exactly solvable statistical mechanics models! Similarly, Yang ran into the equation while trying to cook up 2d quantum field theories for which one could exactly calculate the S-matrix. I won't get into *how* this equation does the trick right now; I just want to note that a whole industry developed of finding solutions to the Yang-Baxter equations. This eventually led to the discovery of quantum groups... but for now I'll leave you with a nice solution of the Yang-Baxter equations: let V be spanned by two vectors X and Y, and define R by

$$\begin{aligned} R(X \otimes X) &= X \otimes X \\ R(Y \otimes Y) &= Y \otimes Y \\ R(X \otimes Y) &= q(Y \otimes X) \\ R(Y \otimes X) &= q(X \otimes Y) + (1-q^2)(Y \otimes X) \end{aligned}$$

Exercise: check that R satisfies the Yang-Baxter equations and $R^2 = (1-q^2)R + q^2$. This quadratic equation is called a Hecke relation.

If you are interested in learning about quantum groups, the Yang-Baxter equations, and braids, I suggest taking a look at the following books:

L. Kauffman, *Knots and Physics,* World Scientific, New Jersey, 1991.

C. Yang and M. Ge, *Braid Group, Knot Theory, and Statistical Mechanics,* World Scientific, New Jersey, 1989.

T. Kohno, *New Developments in the Theory of Knots,* World Scientific, New Jersey, 1990.

Yu. I. Manin, *Quantum Groups and Noncommutative Geometry,* Les Publ. du Centre de Récherches Math., Universite de Montreal, Montreal, 1988.

5 The Quantum Plane

Ever so slowly I'm creeping towards a description of my own research on the relation between braids and quantization. There are so many inviting byways... Anyway, recall where we were last time. (I never knew I'd become so professorial in my old age.)

Given an 1-1 and onto linear transformation $R: V \otimes V \to V \otimes V$, the Yang-Baxter equations say that

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

where I is the identity on V. The most famous example of a solution of the Yang-Baxter equations is where V is spanned by two vectors X and Y, and

$$\begin{aligned} R(X \otimes X) &= X \otimes X \\ R(Y \otimes Y) &= Y \otimes Y \\ R(X \otimes Y) &= q(Y \otimes X) \\ R(Y \otimes X) &= q(X \otimes Y) + (1 - q^2)(Y \otimes X) \end{aligned}$$

Now, given any solution (R, V) of the Yang-Baxter equations, or "Yang-Baxter operator," we can define a "quantum vector space" $S_R V$ as follows. Well, wait — mathematicians like to ladle on the definitions and explain the point later, but I really should give you some clue about "quantum vector spaces" before proceeding.

Quantum vector spaces were basically invented by Manin in his book, "Noncommutative Geometry and Quantum Groups," and I will only be talking about a subclass of *his* quantum vector spaces. To get at the concept, you have to think like an algebraic geometer — or a quantum field theorist. To an algebraic geometer, what counts about a space is the algebra of functions on that space (to be a bit more precise, algebraic functions from that space to the complex numbers... and please, algebraic geometers, don't bother telling me how much I'm oversimplifying — I know). To a field theorist, what counts about spacetime is the fields on spacetime, which are again functions from spacetime to the complex numbers. (And again, I'm vastly oversimplifying things.) There's a common theme here: for the algebraic geometer *points* may be regarded as secondary in importance to *functions*, while for the physicist *particles* may be regarded as secondary in importance to *fields*. This makes a lot of sense when you pay attention to what you're doing as you calculate: most of the time you are playing around with functions, and only at the end (if ever) do you bother to evaluate them at a given point.

Well, regardless of how *true* this actually is, it's a fruitful point of view! It gives rise to noncommutative geometry, in which points are not the point; rather, all attention is concentrated on the algebra of "functions on space," which is allowed to have a noncommutative product. Here I should mention Alain Connes, whose paper "Non-commutative differential geometry" really got the subject rolling in the early 1980's, though the notion of replacing functions on an honest space with a noncommutative algebra is as old as Heisenberg's matrix mechanics. In quantum mechanics, phase space disappears, but the analog of the functions on phase space, the algebra of observables (typically operators on a Hilbert space) lives on.

So now let us define our quantum vector space: what we'll *really* be defining is an abstract algebra, but in the back of our heads we'll always secretly pretend it consists of functions on some (nonexistent) space, a "quantum space."

First, notice that a Yang-Baxter operator R on V gives rise to a Yang-Baxter operator $(R^{-1})^*$ on the dual space V^* — the inverse of the adjoint of R. Recall that V^* is none other than the linear functions on V. Normally we define the algebra of polynomial functions on V, or the "symmetric algebra" SV^*, to be the quotient of the tensor algebra TV^* by the ideal generated by elements of the form

$$f \otimes g - g \otimes f$$

In other words, we take all (formal) products of linear functions on V and impose on them the "commutation relations" that f times g should equal g times f. Note that here we are *switching* f and g; in our quantum vector space we will use the Yang-Baxter operator to do this switching. Thus we define the "r-symmetric quotient of TV^*" by the ideal generated by elements of the form

$$f \otimes g - (R^{-1})^*(f \otimes g)$$

That's all there is to it — the philosophy is that we should never switch two objects blindly in the usual manner, since they might be "anyonic": we should always switch things with the appropriate Yang-Baxter operator.

Now, all these duals and adjoints really just clutter things up a bit; let's just define the r-symmetric algebra $S_R V$ to be TV modulo the ideal generated by

$$v \otimes w - R(v \otimes w)$$

So if we use the example of a Yang-Baxter operator given up above, that I said was so famous, we get the following r-symmetric algebra, called the **quantum plane:** it has coordinates X and Y, which do not commute, but satisfy

$$\boxed{XY = qYX} \quad \textbf{!!!}$$

The less jaded of you will be dazzled by the mere appearance of this bizarre entity. The more jaded will wonder what it's good for. Patience, patience. Let me just say for now that many papers have been written about this thing, in respectable physics journals, so it must be good for something. (?)

One can do all sorts of geometry on the quantum plane just as one does on the good old classical plane. There is a "quantum group" of "quantum matrices" which acts as linear transformations of the quantum plane. One can differentiate and integrate functions on the quantum plane, and so on. In fact, MIT just installed new blackboards in some of the math classrooms which have a dial with which the instructor can select the value of q.

All but the last sentence of the previous paragraph is true. In fact, the simplest quantum group, $SL_q(2)$, consists of 2-by-2 "quantum matrices" with "quantum determinant" equal to 1. It is a prototype for the quantum groups $SL_q(n)$. People are having a ball these days coming up with q-analogs of everything they know about. For exampe, all semisimple Lie groups admit quantizations of this general type, and such quantum groups turn out to arise as symmetries of field theories in 2 or 3 dimensions (where anyonic statistics arise), and to be crucial for understanding the HOMFLY polynomial and other knot invariants. I'm running ahead of myself here and should quit and go to bed; next time I'll either define quantum matrix algebras and $SL_q(2)$, or start on what I'm *really* interested in, the differential geometry of certain "quantum spaces" (more precisely, algebras equipped with Yang-Baxter operators).

6 r-Commutative Geometry

While there are plenty of things to say about the quantum plane, and quantum groups, I think I'll home in on my main topic at last: r-commutative geometry. This is a particular approach to noncommutative geometry that generalizes what mathematicians and physicists call "supergeometry". So first I should say a brief word about supergeometry. First, recall that the plain old geometry of manifolds can be cast into the language of commutative algebra by considering not the manifold itself as a set of points, but the *algebra* of (smooth, complex) functions *on* the manifold. (In this language, for example, vector fields are derivations, vector bundles are projective modules, and so on — every geometric construct has an algebraic analog.) Now in the 70's physicists really caught on to the fact that considering only commutative algebras was horribly unfair to fermions, which like to anticommute: i.e. they have

$$ab = -ba$$

instead of

$$ab = ba.$$

So while the phase space of bosonic system is a manifold, the phase space of a system containing both bosonic and fermionic degrees of freedom is a "supermanifold" — a (particular kind of) algebra which has "even" and "odd" elements, such that the even, or bosonic, elements commute, while the odd elements anticommute. (The even elements commute with the odd elements, by the way.) Generalizing lots of concepts from commutative algebras to supercommutative algebras simply amounts to sticking in appropriate minus signs! The rule of thumb is that whenever one switches two elements a and b, one should stick in a factor of $(-1)^{\deg a \deg b}$, where $\deg a = 0$ if a is even (bosonic) and $\deg a = 1$ if a is odd (fermionic). One may extend the notions of vector field, differential form, metric, curvature, and all your favorite concepts from geometry to "supergeometry" in this manner.

This turned out to be fascinating (it's a bit premature to say "useful") in particle physics, where it goes by the name of supersymmetry. The idea is that there should be a symmetry between bosons and fermions. While this is *not at all* observed in nature, it would be nice if it *were* true, so people have developed clever ways of rigging up their theories so that you never see the "superpartners" every particle has: for the photon, the photino, for the gluon, the gluino, for the leptons, schleptons, for the quarks, squarks... you get the pattern. (Don't complain to *me* if you think this is silly, it wasn't my idea!) Superstrings are the latest of these "super" ideas in particle physics.

Supersymmetry has actually proven itself in a more practical manner in nuclear physics, where it lets one model resonances in nuclei, relating the properties of fermionic and bosonic nuclei.

Where supergeometry really shines, though, is in mathematics. For example, Ed Witten came up with beautiful proofs of the Atiyah-Singer index theorem and the positive mass theorem (a theorem about general relativity) using supergeometry. (As usual, he left it to others to make his arguments rigorous.)

While I personally don't think that bosons and fermions were created equal in the manner postulated by supersymmetry, I do favor an approach to physics which doesn't take bosons, or commuting variables, to be somehow superior to fermions, or anticommuting variables. This demands supergeometry. For example, a decent treatment of *classical* fermions requires a supermanifold for the "phase space".

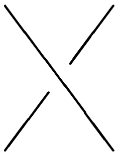
My own personal twist (motivated by the work of many people on anyons, the braid group, quantum groups, etc.) is to try to take a look at what geometry would be like if one wanted to be fair to *anyons* as well as bosons and fermions. This is "r-commutative geometry."

Recall that given a vector space V, an invertible linear transformation

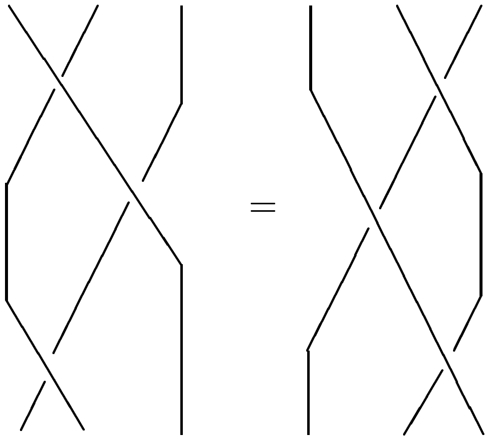
$R : V \otimes V \to V \otimes V$, the Yang-Baxter equations say that

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

where I is the identity on V. The idea is that R "switches" two elements of V, mapping $v \otimes w$ to $R(V \otimes w)$, and if we draw R as a "crossing," as follows:



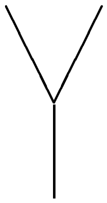
The Yang-Baxter equations say that



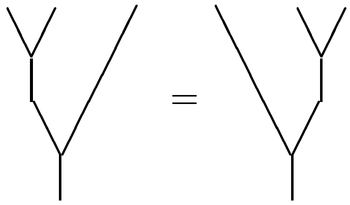
Now suppose that our vector space is really an algebra — let's call it A. (I mean an associative algebra with unit.) The product in the algebra defines a multiplication map $m : A \otimes A \to A$, given by

$$m(a \otimes b) = ab.$$

We can draw this as the *joining* of two strands:



Associativity simply says that $(ab)c = a(bc)$, or in terms of diagrams,

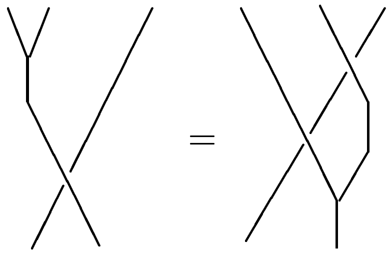


Now these diagrams aren't really braids because of the "fusion of strands" that's taking place, but they fit in well with the braid group philosophy. For example, there is a sense in which the two trees I've drawn above are topologically the same, just as the Yang-Baxter equation expresses a topological identity. There are generalizations of braids, e.g. the "ribbon graphs" of Reshetikhin and Turaev, that make all this precise.

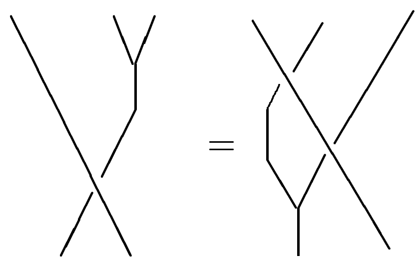
Now: an "r-algebra" is an algebra A equipped with a solution $R : A\otimes A \to A \otimes A$ of the Yang-Baxter equations such that:

1) $R(a\otimes 1) = 1\otimes a$ and $R(1\otimes a) = a\otimes 1$ for all a in A — i.e., one "switches" the identity 1 with a in the usual manner;

2) the following conditions hold — I'll draw them pictoriallly:



and



If you like equations instead of pictures, these are

$$R(m \otimes I) = (I \otimes m)(R \otimes I)(I \otimes R)$$

and

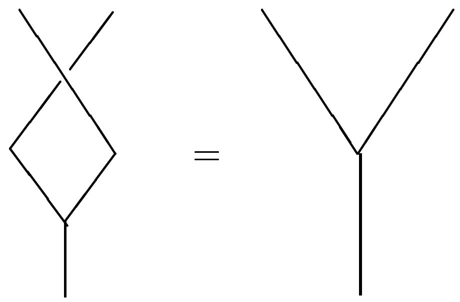
$$R(I \otimes m) = (m \otimes I)(I \otimes R)(R \otimes I),$$

as maps from $A \otimes A \otimes A$ to $A \otimes A$, respectively. The first one tells you how to compute $R(ab \otimes c)$, and the second one tells you how to compute $R(a \otimes bc)$, so both tell you how to switch a product of two elements past a third element. These are called the "quasitriangularity" conditions and are crucial in the theory of quantum groups (usually in a slightly disguised form).

Now commutativity says that $ab = ba$ — in other words, you can multiply two elements of A, or you can switch them first and then multiply them, and you'll get the same result. Generalizing this, we say that an r-algebra is "r-commutative" if

$$m = mR$$

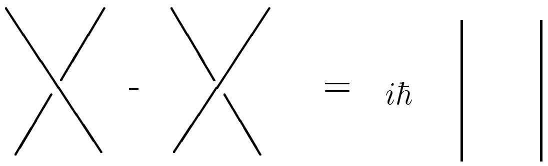
as maps from $A \otimes A$ to A. In diagrams:



In words: "switch, then multiply, equals multiply."

It turns out that one can do a fair amount of geometry for r-commutative algebras. And there are lots of examples of r-commutative algebras. In fact, many of the algebras obtained by *quantization* are r-commutative: for example, the Clifford algebra, the Weyl algebra, noncommutative tori, the quantum plane (and all other r-symmetric algebras), and quantum groups. Also, all supercommutative algebras are automatically r-commutative.

In fact, I believe that there is a deep relation between braids (or r-commutativity) and quantization. This goes back to Kauffman's observation that the skein relation for the Conway polynomial



and the canonical commutation relations

$$pq - qp = -i\hbar$$

It was made more clear by the discovery of the relation between quantum groups and knot invariants (which I haven't really touched upon yet — here I highly recommend Louis Kauffman's book *Knots and Physics*, by World Scientific Press). It was made still more clear (in my opinion) by the work of Frohlich, Gabbiani, Rehren, Schroer, and many others on the appearance of braid group statistics in low-dimensional quantum field theory. (Essentially, these authors show that every nice quantum field theory gives rise to a "fusion algebra" for the conserved charges, and that these fusion algebras are r-algebras.) But I suspect that at the root of it all is something rather simpler which we haven't understood yet. A clue, I think, lies in the "classical Yang-Baxter equations." These are what physicists would call a "semiclassical limit" of the Yang-Baxter equations. Namely, take $R: V \otimes V \to V \otimes V$ of the form $R = T(1 + \hbar r)$, where T is the usual twist map

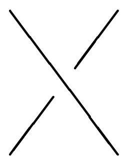
$$T(v \otimes w) = w \otimes v$$

and $r : V \otimes V \rightarrow V \otimes V$, write down the Yang-Baxter equation for R, and collect all terms of order $\hbar^2$. This equation is identical to the equation that a "Poisson structure" must satisfy. For details, see Drinfeld's famous review paper on quantum groups. The point here is that a Poisson structure, which defines the Poisson bracket of classical observables, is a semiclassical limit of the commutator which appears in quantum mechanics. Thus an infinitesimal deformation of the usual twist map that is required to satisfy the Yang-Baxter equation is *the same thing* as an infinitesimal deformation of the commutative product in the classical algebra of observables (i.e., a Poisson bracket).

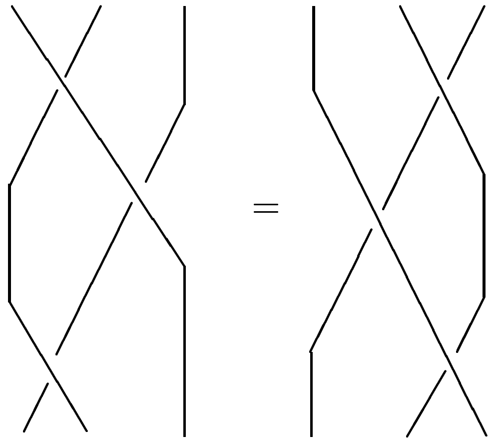
More later...

7 r-Algebras

So...let's say we have an r-algebra. That's an algebra with an invertible linear map $R : A \otimes A \rightarrow A \otimes A$, which we draw as a right-crossing:

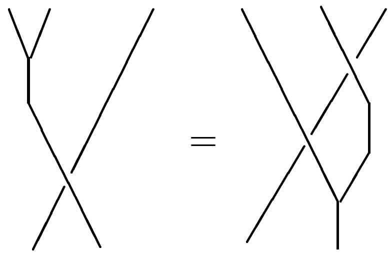


which satisfies 1) the Yang-Baxter equations:

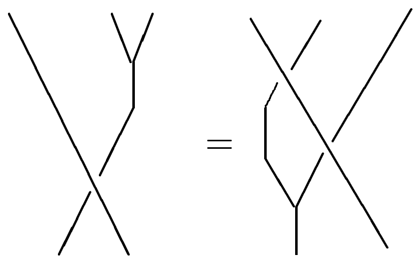


2) $R(1 \otimes a) = a \otimes 1$ and $R(a \otimes 1) = 1 \otimes a$,

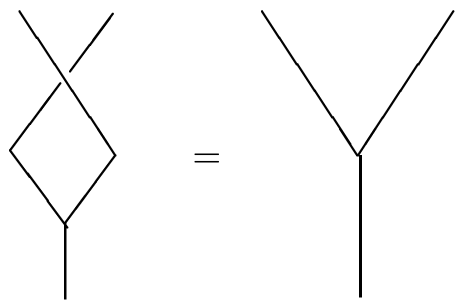
and 3) the quasitriangularity conditions; writing multiplication as the joining of strands these are



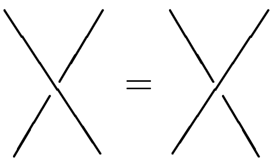
and



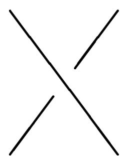
We say an r-algebra is "r-commutative" if



and "strong" if R^2 is the identity, or R equals its inverse, i.e.:



(Note that a left-crossing



is the inverse of a right-crossing.)

All sorts of noncommutative analogs of manifolds are r-commutative algebras: quantum groups, noncommutative tori, quantum vector spaces, the Weyl and Clifford algebras, certain universal enveloping algebras, supermanifolds, etc. It seems that the ones with direct relevance to quantum theory in 4 dimensions are "strong," while the non-strong ones, like quantum groups, are primarily relevant to 2- and 3-dimensional physics. I would now like to describe an analog of differential forms for strong r-commutative algebras, and illustrate it for the case of the Heisenberg algebra — i.e., the algebra defined by the canonical commutation relations: $pq - qp = -i\hbar$.

What are differential forms? Of course, they're the basis of a lot of differential geometry, and there are lots of equivalent ways of defining them, but let me take an algebraic viewpoint. Let A be the algebra of smooth functions on a manifold M. We define differential forms as follows. Each function f in A has a "differential" df, and the functions and their differentials generate an algebra in which we impose the following relations:

1. Linearity: $d(f+g) = df + dg$, $d(cf) = c(df)$ for any scalar c.
2. The product rule: $d(fg) = (df)g + f(dg)$.
3. The derivative of a constant vanishes: $d1 = 0$.
4. Commutation relations: $f(dg) = (dg)f$ and $(df)(dg) = -(dg)(df)$.

This algebra is called the algebra of differential forms on M.

That's all, folks! If you've taken a course in differential geometry you were probably exposed to tangent planes and all that stuff, but if you want to get calculating with differential forms as soon as possible this is all you need to start with.

Note that rule 4) is the only one in which we *switch* elements of A — moving f to the right of g. This is the rule we'll need to modify for an r-commutative algebra. You could, of course, just leave out rule 4): given any algebra A, the algebra whose relations are just given by 1)–3) is called the "universal differential calculus" for A. It's a reasonable substitute for differential forms when A is any old noncommutative algebra, and (from one viewpoint) it's the basis of Alain Connes' approach to noncommutative differential geometry. But when you have a strong r-commutative algebra one can replace rule 3) with

4′) Commutation relations: $f(dg) = (dg^i)f_i$ and $(df)(dg) = -(dg^i)(df_i)$.

Here I should explain that I'm writing $R(f \otimes g)$ as the sum over i of tensor products of the form $g^i \otimes f_i$ (one can always do this), and I'm using the Einstein summation convention (sum over repeated indices) to avoid writing the summation sign. What we're doing in rule 4′) is just what we should do: use R to "switch" f and g instead of "naively" switching them as in 4).

Now let me show what this buys you in the case of the Heisenberg algebra. This is the algebra over C generated by 3 formal variables, p, q, and h, subject to the relations that

$$pq - qp = -ih^2$$

$$ph = hp$$

$$qh = hq$$

Note that we're *not* treating h as a number here, but as a variable, so I have to explicitly *say* that it commutes with everything. (If h *were* a number, it would be $\sqrt{\hbar}$.)

This algebra is actually a strong r-commutative algebra in a unique manner such that

$$R(p \otimes h) = h \otimes p$$

$$R(q \otimes h) = h \otimes q$$

and

$$R(p \otimes q) = q \otimes p - ih \otimes h$$

Note what these say: the first two say that you switch h with p and q in the usual way, and the third one says that when you switch p and q, you get the expected term *and then* a piece, $-ih \otimes h$, which comes from the fact that p and q don't commute. The point is that while the Heisenberg algebra is not commutative, the canonical commutations relations do tell you exactly what to do when you switch p and q, which is just as good!

If I may digress... I happen to have the paper by Heisenberg, Born, and Jordan with me, "Zur Quantenmechanik. II.", published in 1926, in which the canonical commutation relations are introduced. I quote:

> Das Rechnen mit den quantentheoretischen Grössen würde wegen der Nichtgültigkeit des kommutativen Gesetzes der Multiplikation in gewissem Sinne unbestimmt bleiben, wenn nicht der

> Wert von $pq - qp$ vorgeschrieben würde. Wir führen daher als fundamentale quantenmechanische Relation ein:[1]
>
> $$pq - qp = (\frac{h}{2\pi i})\mathbf{1}.$$

If we take the Heisenberg algebra as a strong r-commutative algebra and define the differential forms on this algebra by the rules 1), 2) and 3'), we get the following standard-looking relations:

$$\begin{array}{llll} h\,dp = dp\,h, & h\,dq = dq\,h, & p\,dh = dh\,p, & q\,dh = dh\,q, \\ h\,dh = dh\,h, & p\,dp = dp\,p, & q\,dq = dq\,q & \end{array}$$

and a bunch of similar ones involving two differentials:

$$\begin{array}{llll} dh\,dp = -dp\,dh, & dh\,dq = -dq\,dh, & dp\,dh = -dh\,dp, & dq\,dh = -dh\,dq, \\ dh\,dh = 0, & dp\,dp = 0, & dq\,dq = 0 & \end{array}$$

but then some more interesting ones:

$$\begin{gathered} p\,dq - dq\,p = -ih\,dh \\ q\,dp - dp\,q = ih\,dh \\ dp\,dq = -dq\,dp \end{gathered}$$

(It's a nice little exercise to see that these really *do* follow from 1)-4$'$)) One can have fun doing various things with these "quantized differential forms" (and their generalizations), basically by taking your favorite facts from the differential geometry of phase space and trying to "quantize" them, but let me just briefly run through the basics. (Now I will let myself be more mathematical.) One may form a quotient of the Heisenberg algebra by specializing the variable h to some value; this is called the Weyl algebra. The corresponding differential forms on the Weyl algebra were discovered by I. Segal some time ago and called "quantized differential forms;" their cohomology gives a nice way of understanding the Wick product in quantum field theory.

[1] "Due to the failure of the commutative law for multiplication, computations with quantum-theoretical quantities are ambiguous in a certain sense, unless one prescribes a value for $pq - qp$. We introduce accordingly the following as the fundamental quantum-mechanical relation: [etc.]"

(He also dealt with a fermionic version using the Clifford algebra.) In my paper on r-commutative geometry I recommend an approach in which one views the Heisenberg algebra as a "bundle" over the "line" whose coordinate is given by h. A given "fiber" (at which h has some numerical value) is then a Weyl algebra (except for the "classical fiber" at $h = 0$, which is just the algebra of polynomials on phase space). I put quotes around the words "fiber bundle" because only the classical fiber is really a manifold; the rest are "quantum manifolds," i.e. noncommutative algebras. But the fiber bundle viewpoint gives good insight into the relation between the differential forms on the base (the line), the fibers (Weyl algebras) and the total space (the Heisenberg algebra). This viewpoint also works for noncommutative tori (and other cases). Neither of these examples are exciting from the point of view of braid invariants, since they are "strong". The differential forms as defined above do not work very well for non-strong r-algebras. I have no idea whether there is a good general definition of differential forms for r-commutative algebras; there is a definition that works for the quantum plane and other quantum vector spaces with Hecke-type relations (i.e., not $R^2 = 1$, but $R^2 = (1-q)R + q$.) I just finished a paper, "Hochschild Homology in a Braided Tensor Category," in which I define a generalization of Hochschild homology for r-algebras and relate it to the r-commutative differential forms defined above. (I have not yet computed this homology theory in non-strong cases.) This paper should appear in Transactions of the American Mathematical Society shortly after hell freezes over. In the meantime, you may content yourself with: R-commutative geometry and quantization of Poisson algebras, Adv. Math. **95** (1992) 61 - 91.

8 Noncommutative Tori

I'm beginning to tire out but there is one loose end (out of many) that I'd like to nail down. I've mentioned noncommutative tori a couple of times but haven't defined them or said what they have to do with physics.

Okay: recall that $L^2(\mathbf{R})$ means the Hilbert space of square-integrable complex function on the real line. If we define the unitary operators u and v on $L_2(\mathbf{R})$ given by

$$\begin{aligned} u &= \text{translation by the amount } s \\ v &= \text{multiplication by } e^{ix} \end{aligned}$$

we can see that they don't commute, but instead satisfy

$$uv = qvu$$

where $q = e^{is}$. (Note: I didn't say *which way* to translate by the amount s — if you pick this correctly things will work out.)

The algebra of operators on $L^2(\mathbf{R})$ generated by u and v and their inverses is called the noncommutative torus T_q. (If you know how, it's better to take the C*-algebra generated by two unitaries u and v satisfying

$$uv = qvu.$$

This is actually quite a bit bigger for $q = 1$.) This is clearly a natural sort of thing because it's built up out of simple translation and multiplication operators, and all of Fourier theory is based on the interplay between translation and multiplication operators.

Why is called a "torus"? Note that it depends on the parameter q. If we take $q = 1$, T_q is the C*-algebra generated by two unitaries u and v that *commute*. This may identified the algebra of functions on a torus if we think of u as multiplication by $e^{i\theta}$ and v as multiplication by $e^{i\phi}$, where θ and ϕ are the two angles on the torus. So we've got a one-parameter family of algebras T_q such that when $q = 1$, it's just the algebra of (continuous) functions on a torus, but for q not equal to one we have some sort of noncommutative analog thereof. The parameter q measures noncommutativity or "quantum-ness", and one can relate it to Planck's constant (which also measures "quantumness") by

$$q = e^{i\hbar}$$

This example is actually the tip of an iceberg called called "deformation theory". One can read more about it Marc Rieffels' paper "Deformation quantization and operator algbras," Proc. Symp. Pure Math. 51, or (from a different viewpoint) "Deformation theory and quantization" by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer, in Ann. Physics 111, p. 61–151. As you can see from these titles, it has a lot to do with quantization, since in quantization one is trying to start with a commutative algebra of functions (observables) on phase space and "quantize" it, or make it noncommutative, somehow.

Now it shouldn't be too surprising that the noncommutative torus is an r-commutative algebra, since the commutation relations $uv = qvu$ tell you exactly how to "switch" u and v. I've shown that there is a unique way to make the noncommutative torus into a strong r-commutative algebra such that

$$R(u \otimes u) = u \otimes u$$

$$R(v \otimes v) = v \otimes v$$

and

$$R(u \otimes v) = qv \otimes u.$$

(Recall that "strong" simply means that R^2 is the identity, so $R(v \otimes u) = q^{-1}u \otimes v$.) One may thus go ahead and define "r-commutative differential forms" for the noncommutative torus, which satisfy

$$\begin{array}{ll} u(du) = (du)u, & v(dv) = (dv)v, \\ u(dv) = q(dv)u, & (du)v = qv(du) \end{array}$$

and more relations obtained by differentiating these. One can then calculate the (r-commutative) de Rham cohomology of the noncommutative torus, and, lo and behold, it's isomorphic to that of the usual torus. This fits into the philosophy that the noncommutative torus is obtained from the usual torus by a "continuous deformation" — no holes have been formed or gotten rid of.

If you're interested in learning more about noncommutative tori, a good review article is "Noncommutative tori: a case study of noncommutative differential manifolds," Contemp. Math. 105, p. 191, by Rieffels. If you're interested in the r-commutative geometry of noncommutative tori, try my "R-commutative geometry and quantization of Poisson algebras," which will appear in Adv. Math..

Let me just conclude by saying that noncommutative tori have applications in physics. This shouldn't really be surprising since they're such simple things. Let's say you have a charged particle trapped on the xy plane, and there's a magnetic field of constant intensity B perpendicular to the plane. Then the momentum operators in the x direction and the y direction no longer commute. Exponentials of these (i.e., translations) generate a noncommutative torus, and this fact has been used by Belissard to do certain calculations of the quantum Hall effect! See J. Bellisard, *K-theory of C*-algebras in solid state physics*, Springer Lecture Notes in Physics 257.

9 Charged Particles in a Magnetic Field

I've decided that the end of my previous article is just a bit too brief and enigmatic for those not up on their quantum mechanics. So I'll explain what happens to a charged particle in the plane when its in a magnetic field, and how one gets a noncommutative torus out of this situation. Then, since it's the holiday season, I'll let myself digress to discuss a particle on a *sphere* in a magnetic field.

First, let's set $\hbar = 1$. *CLICK!*

Suppose we have a particle on the plane — with no magnetic field. Then in quantum mechanics the momenta in the x and y directions are given by the operators,

$$p_x = -i\partial/\partial x \quad \text{and} \quad p_y = -i\partial/\partial y$$

respectively. These commute, becuase mixed partials commute.

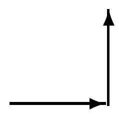
Now let's turn on the magnetic field pointing perpendicular to the plane. Let's say our particle has charge $= 1$, and the field is $\mathbf{B}$ (of magnitude B). The curious fact about quantum theory is that (if we neglect the *spin* of the particle) the only effect of the magnetic field is to make us redefine the momentum operators to be

$$p_x = -i\partial/\partial x + A_x \quad \text{and} \quad p_y = -i\partial/\partial y + A_y$$

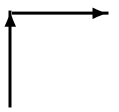
where $\mathbf{A}$, the vector potential, has $\nabla \times \mathbf{A} = \mathbf{B}$. Now p_x and p_y don't commute, and in fact the commutator

$$p_x p_y - p_y p_x$$

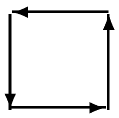
is just $-iB$ . . . as everyone should check for whom it isn't instantly, blindingly obvious. It's a pity that when I was first learning quantum mechanics the teacher didn't remark on how curious and charming this is: when there's a magnetic field around, the different components of the momentum no longer commute, and the amount by which they fail to commute is precisely the magnetic field! Recall that the meaning of momentum in quantum theory is that it's the generator of spatial translations. This means that if you grab your charged particle and move it first along the x direction and then the y direction:



the particle winds up in a different state than if you first go in the y direction and then the x direction:



Another way of thinking of it is as follows: take your particle, move it counterclockwise (say) around a rectangle:



and it'll be back in the same place, but its wavefunction will not be the same as it was: it'll differ by a phase (multiplication by a complex number of unit magnitude). It's easy to calculate that if we call the particles wavefunction to start out with "ψ," and when we're done "ϕ,"

$$\phi = e^{is}\psi$$

where s is just the line integral of $\mathbf{A} \cdot d\mathbf{l}$ around the rectangle routes. By Stokes' theorem, s is just the integral of B over the rectangle!

If I had been told this, I might have absorbed it a bit more quickly when I was told in fancier language later on that the momentum operators p_x and p_y were components of a "connection" on a "complex line bundle," and that their commutator was the "curvature" of this connection, so that the magnetic field is really a curvature, and that the difference in phase obtained by taking two routes is called the "holonomy" of the connection. For that's the modern way of discussing this sort of thing.

Anyway, now suppose that the magnetic field strength is a constant B. Let U denote the unitary operator corresponding to translation by a unit distance in the x direction, and let V be the unitary operator corresponding to a unit translation in the y direction. Then by what I've said, we have

$$UV = qVU$$

where $q = e^{iB}$. Thus U and V satisfy the relations of a noncommutative torus.

More generally, if you move a charged particle counterclockwise around any loop, its phase changes by e^{is}, where s is the integral of B over the region enclosed by the loop. Note that the "counterclockwise" bit comes from using Stokes' theorem. (Also, an oppressive majority of right-handed people have enforced foolish "right-hand rules" ...don't you think it'd make more sense to call *clockwise* the positive orientation? Oh well.) If we go clockwise, the phase change is e^{-is}.

This still holds when our particle is not on a plane but on some other sort of two-dimensional surface, but there are some curious consequences.

Consider, for example, a particle on a sphere (for mathematicians, S^2), with a magnetic field applied that's normal to the sphere at each point. If we move the particle around a loop the phase change will equal the integral of the magnetic field over the region enclosed by the loop. But wait a minute! There are two different regions, the "inside" and the "outside" of the loop, which count as regions enclosed by the loop! (Just draw a circle on a sphere and look!) Who is to say which one we should use to calculate the phase change? There's only one way out: we had better get the same answer each way! That is, if we call the integral of the magnetic field over the region "inside" the loop B_1, and over the region "outside" the loop B_2, we must have

$$e^{iB_1} = e^{-iB_2}$$

Where'd that minus sign come from? Well, if the loop goes counterclockwise around one region, it goes clockwise around the other region, so one of them gets a minus sign. (Draw a circle on a sphere and look!) In any event, if we write $B = B_1 + B_2$ for the integral of the magnetic field over the whole sphere, the above equation gives

$$e^{iB} = 1$$

so B must be an integer multiple of 2π!

This is an odd but true result, and it applies not only to the sphere but to any (compact, oriented) surface: we can only make sense of a charged quantum mechanical particle on such a surface in a magnetic field perpendicular to the surface if the integral of the magnetic field is a integer multiple of 2π.

This result too may be gussied up in fancy mathematical language. (And it's not just jargon, but crucial in understanding this problem more deeply.) The wavefunction of our charged particle on the sphere, or any other surface, is not really a function, but a section of a complex line bundle over the surface. Giving such a line bundle a connection, calculating the curvature, integrating the curvature over the surface, and dividing by 2π, we must get an integer! (Again, I'm assuming the surface is compact and oriented: e.g. a sphere, torus, or donut with more holes...) This integer is called the first Chern number c_1 of the line bundle. A deeper theorem says that given a surface we can cook up a line bundle with any desired c_1, and another theorem says that line bundles over surfaces are completely classified by their first Chern number.

Okay, now — did you notice the following physical problem with what I did? Say we have a sphere in R^3, and a magnetic field perpendicular to the sphere, such that the integral over the sphere is $2\pi n$. (As we saw above n has to be an integer.) Unless $n = 0$, this means that there is a net magnetic flux flowing in or out of the sphere — which contradicts the fact that $\nabla \cdot \mathbf{B} = 0$! I.e., Gauss' theorem says that the integral of the normal component of $\mathbf{B}$ over the sphere is 0, since $\nabla \cdot \mathbf{B} = 0$.

Well, this didn't bother Dirac! In fact, he came up with all this stuff when he was studying magnetic monopoles! He considered the possibility that $\nabla \cdot \mathbf{B}$ was nonzero in the vicinity of some monopole, but considered a big sphere around the monopole, assumed $\nabla \cdot \mathbf{B} = 0$ in the vicinity of this big sphere, and used quantum mechanics to show (as above) that total magnetic charge inside the sphere must be an integer multiple of 2π! I've dropped all the units, but if you throw $\hbar$ and the electron charge back in correctly, you get a "quantum of magnetic charge".

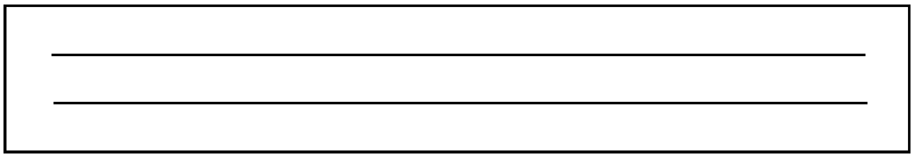
There's another way around the problem, too, which mathematicians can tolerate, if not physicists: just say "I'm considering a particle in a magnetic field on some *abstract* S^2, not one sitting inside $\mathbf{R}^3$, so I don't need to worry about the ball 'inside' my sphere."

Or, if you like, you can say, "How do you know there's not a *wormhole* inside my sphere, so that the $\mathbf{B}$ field can be pouring in from somewhere else?" In other words, "My sphere is a nontrivial 2-cycle in a 3-manifold, so it is possible for a closed 2-form (the magnetic field) to have nonzero integral over it." This is the basis for a suggestion by Wheeler, that charged particles are really the mouths of wormholes, and that actually the divergence of $\mathbf{E}$ and $\mathbf{B}$ are zero everywhere.

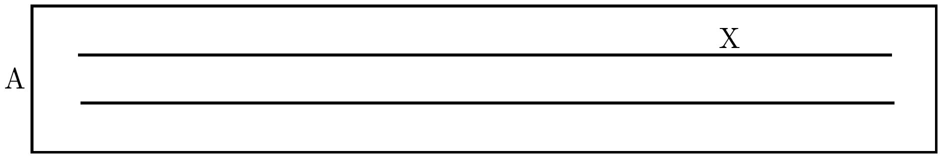
As you can see, weaseling out of this problem can take many interesting forms! That is, perhaps, the essence of mathematical physics. :-)

10 Leatherworking

Okay, this is a "just-for-fun" posting on braids. I had a mind-bending exploration of topology the other night with a friend that I'd like to recommend to all of you. Take a long thing piece of paper in it and cut two parallel long slits in it like so:



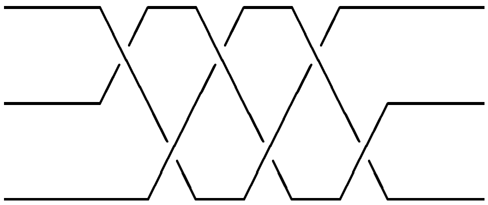
This represents the trivial braid on three strands. You can actually get nontrivial braids by "cheating" and grabbing the end marked A (below) and sliding it through one of the slits — say the top one, around the position marked X (the exact position doesn't matter, of course), and then pulling it back to about its original position:



If you do this in the simplest possible way, you will get a braid in which two of the strands cross around each other twice, while the third strand is not tangled with the other two — but all the strands have a 360 degree twist in them now!!

(So really we are working here not with braids but "framed braids," in which each strand has a certain twist to it. Framed braids also form a group which has the ordinary braid group as a quotient.)

Okay, here's your puzzle. Get your piece of paper to look like this as a braid — with no strand having any twist in it:



This sort of braid, where top and bottom strand take turns going over the middle strand, is the typical braid found in hairdos. Here however the exact number of crossings counts. Note two neat things about this braid. First, each strand winds up in its original position (top to top, middle to middle, bottom to bottom) - i.e. its image in the symmetric group is the identity. Second, if we get rid of any one strand the remaining two are unlinke (i.e. form a trivial braid on two strands). Thus it's a braid analog to the "Borromean rings" (three linked circles no pair of which are linked).

Anyway, getting your piece of paper to look like this without any cutting and pasting is a topological trick well-known to leather-workers, who can make seamless leather braids this way. My friend and I were unable to make this braid except using the following trick. Grab the strands near the left (as in the first picture) and braid them to look like the desired braid, ignoring the fact that near the right things are getting all screwed up. Now look at what you have at the right — the inverse braid of the one you want (no surprise, since the whole braid is still the identity braid)! While preserving the left half, which is the way you want it, now use "cheating" moves on the right half (i.e., grab the right end and slip it through the slits) to kill off the unwanted junk (the inverse braid of the one you want). You can do it with three, or perhaps even just two, "cheating moves" — if you're clever! You are now left with the desired braid as in the third picture!

Now there has got to be a more straightforward way of doing this! One should simply be able to create the desired braid by 2 or 3 cheating moves. Unfortunately my friend and I never succeeded. It's sort of like we knew how to differentiate but not how to integrate. But we learned some interesting topology in the process — and that's what counts! So I strongly recommend that everyone make a 2-slitted strip of paper (leather would be better) and see what kinds of framed braids they can make. There is clearly an interesting sort of group lurking here: the subgroup of framed braids that can be generated by "cheating moves". I am sure that topologists have figured this stuff out already, but it's more fun to mess with it yourself in this case.