Higher Gauge Theory,
Homotopy Theory and n-Categories

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Abstract

These are rough notes for four lectures on higher gauge theory, aimed at explaining how this theory is related to some classic themes from homotopy theory, such as Eilenberg–Mac Lane spaces. After a brief introduction to connections on principal bundles, with a heavy emphasis on the concept of 'torsor', we describe how to build the classifying space BG of a topological group G starting from the topological category of its torsors. In the case of an abelian topological group A, we explain how this construction can be iterated, with points of B^nA corresponding to 'finite collections of A-charged particles on S^n'. Finally, we explain how B^nA can be constructed from the n-category of n-torsors of A. In the process, we give a quick introduction to some simple concepts from n-category theory. References provide avenues for further study.

1 A Taste of Gauge Theory

Gauge theory describes the forces of nature using the mathematical formalism of *connections on principal bundles*, which physicists call 'gauge fields'. We will not explain how this works — our goal is instead to explain how principal bundles and their categorified generalizations relate to some basic themes in homotopy theory — but a taste of the original physics motivation will still be helpful. The easiest example is gravity. A physical object can be used to define a 'frame' in the n-dimensional smooth manifold M representing spacetime:

Definition 1. *A* **frame** *at a point x in some smooth manifold M is a basis of the tangent space T_xM. The set of all frames at x is denoted F_xM. The set of all frames at all points of M is denoted FM, and called the* **frame bundle** *of M.*

Ignoring the fourth dimension (time), the picture looks like this:

The frame bundle of M can be made into a smooth manifold, and the motion of a freely falling nonrotating object traces out a path in the frame bundle:

More generally, we may carry an object without rotating it along any smooth path in spacetime — this is called 'parallel transport'. Parallel transport along a smooth path γ from $x \in M$ to $y \in M$ gives rise to a map

$$\mathrm{hol}(\gamma) \colon F_x M \to F_y M$$

called 'the holonomy along γ'. It's easy to visualize: we just imagine carrying a basis of tangent vectors from x to y, doing our best not to rotate it or otherwise mess with it:

This is easy in Euclidean $\mathbb{R}^n$, but it is more tricky when M is a more general manifold. Mathematically, we compute $\mathrm{hol}(\gamma)$ using an extra structure called a 'connection on the frame bundle'. Our goal in this lecture is to explain how this works.

First: what's the mathematical structure of $F_x M$? Given a basis of the tangent space at x, say $f \in F_x M$, and an invertible $n \times n$ real matrix, say $g \in \mathrm{GL}(n)$, we can apply g to f and get a new basis. We write this as fg, since a basis of $T_x M$ is really just a linear isomorphism

$$f \colon \mathbb{R}^n \xrightarrow{\sim} T_x M$$

and composing this with

$$g \colon \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$$

gives a new basis

$$fg \colon \mathbb{R}^n \xrightarrow{\sim} T_x M.$$

So, the group $\mathrm{GL}(n)$ acts on F_xM; by convention it acts on the right. So, wesay F_xM is a 'right G-set'. Furthermore, given any two bases $f, f' \in F_xM$ there exists a unique $g \in \mathrm{GL}(n)$ with $f' = fg$.

We summarize this structure by saying that F_xM is a '$\mathrm{GL}(n)$-torsor':

Definition 2. *Given a group G, a* **G-torsor** *is a nonempty set T equipped with a right G-action such that for every $t, t' \in T$ there exists a unique $g \in G$ with $t' = tg$.*

Equivalently, but even better:

Proposition 3. *Given a group G, a G-torsor is a set T equipped with a right G-action such that T is isomorphic as a right G-set to G itself. In other words, there exists an invertible map*

$$\phi \colon T \xrightarrow{\sim} G$$

with

$$\phi(tg) = \phi(t)g$$

for all $t \in T$, $g \in G$.

The reader should prove this and see why the nonemptiness condition is required. The idea is simple: if T is a G-torsor, for any $t_0 \in T$ we get an isomorphism of right G-sets $\phi \colon T \to G$ that sends $t_0 \in T$ to $1 \in G$. So, we say:

A torsor is a group that has forgotten its identity!

Picking any point as identity lets us think of our G-torsor as being the group G:

As we'll soon see, the holonomy of a connection on the frame bundle:

$$\mathrm{hol}(\gamma) \colon F_xM \to F_yM$$

is a G-torsor morphism. So, we'd better define this notion:

Definition 4. *Given G-torsors T and T', a* **G-torsor morphism** *$\phi \colon T \to T'$ is a map of right G-spaces, that is, a map with*

$$\phi(tg) = \phi(t)g$$

for all $t \in T$, $g \in G$.

If G is a topological group (or Lie group), its torsors naturally become topological spaces (or smooth manifolds), and these are the cases we'll be most interested in. In these cases we demand that our G-torsor morphisms be continuous (or smooth).

We're now ready to define connections on bundles and compute their holonomies!

Definition 5. *Given a topological (or Lie) group G and a topological space (or manifold) M, a* **principal G-bundle over** *M is a topological space (or manifold) P equipped with a right action of G, together with a map*

$$\pi \colon P \to M$$

which is **locally trivial***: any point $x \in M$ has a neighborhood U such that*

$$\begin{tikzcd}
\pi^{-1}U \arrow[rr, "t"] \arrow[dr, "p"'] & & U \times G \arrow[dl, "(u,g)\mapsto u"] \\
& U &
\end{tikzcd}$$

commutes, where t is an isomorphism of right G-spaces, Here $U \times G$ has been made into a right G-space in the obvious way.

Here's a picture of a principal G-bundle, which is strictly accurate when G is the circle group, $\mathrm{U}(1)$:

So, a principal G-bundle over M looks like a bundle of copies of G sitting over the points of M. However, these copies of G are really just G-torsors, since they don't have a chosen identity element:

Proposition 6. *Given a principal G-bundle $\pi\colon P \to M$, each* **fiber** *$P_x = \pi^{-1}x$ is a G-torsor.*

Here's my favorite example of a principal bundle:

Proposition 7. *The frame bundle*

$$FM = \bigcup_{x \in M} F_x M$$

of an n-dimensional smooth manifold M is a principal $GL(n)$-bundle, where $\mathrm{GL}(n)$ *acts on the right thanks to the fact that each fiber $F_x M$ is a* $\mathrm{GL}(n)$*-torsor.*

Now for connections. Here we need everything to be smooth:

Definition 8. *Suppose G is a Lie group and $\pi\colon P \to M$ is a principal G-bundle where P and M are smooth manifolds and π is a smooth map. Then a* **connection** *on this principal G-bundle is a smoothly varying choice of subspaces*

$$H_p \subseteq T_p P$$

that is preserved by the action of G, and such that

$$T_p P = \ker(d\pi)_p \oplus H_p.$$

We call vectors in H_p **horizontal vectors** and call vectors in the kernel of $(d\pi)_p$ **vertical vectors**. A picture explains why:

Given $p \in P$ with $\pi(p) = x \in M$, a connection allows us to lift any smooth path $\gamma\colon [0,1] \to M$ to a unique smooth path $\tilde{\gamma}$ in P once we choose the starting point $\tilde{\gamma}(0)$ to be any desired point sitting over $x \in M$, and require that $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$:

We call $\tilde{\gamma}$ a **horizontal lift** of γ because its tangent vector is always horizontal. In the case of a connection on the frame bundle, this condition says that we are moving the frame along while changing it as little as possible — parallel transporting it, in other words. But, *we need the connection H to know what 'changing as little as possible' means!*

Homotopy theorists are fond of fibrations. These satisfy a 'path lifting condition'. A principal bundle is a fibration so it satisfies this sort of condition. What a connection does is pick out a *specific way* to lift any path γ in M to a path in P, given a lift of the starting point $\gamma(0)$. This is the horizontal lift $\tilde{\gamma}$.

Suppose that γ starts at $x \in M$ and ends at $y \in M$. Since γ has a unique horizontal lift $\tilde{\gamma}$ after we pick the starting point $\tilde{\gamma}(0)$ to be any point in the fiber over x, we obtain a map

$$\begin{array}{rcl} \mathrm{hol}(\gamma)\colon P_x & \to & P_y \\ \tilde{\gamma}(0) & \mapsto & \tilde{\gamma}(1) \end{array}$$

called the **holonomy of the connection H along the path γ**. It looks like this:

Since the connection is G-invariant, one can show:

Proposition 9. $\mathrm{hol}(\gamma)\colon P_x \to P_y$ *is a G-torsor morphism.*

Here are some exercises to help you master the all-important concept of torsor:

Exercise 10. *Write down an equivalent definition of a G-torsor as a set T equipped with a 'multiplication'*

$$\begin{array}{rcl} T \times G & \to & T \\ (t,g) & \mapsto & tg \end{array}$$

and a 'division'

$$\begin{array}{rcl} T \times T & \to & G \\ (t,t') & \mapsto & \frac{t'}{t} \end{array}$$

such that

$$t\,\frac{t'}{t} = t'.$$

What other axioms are needed?

Exercise 11. *Show that the indefinite integral of a real function of one variable, $\int f(x)dx$, is actually an $\mathbb{R}$-torsor. Here we treat $\mathbb{R}$ as a group with addition as the group operation. (Hint: this explains that annoying '$+C$' from freshman calculus.)*

Exercise 12. *Show a* $\mathrm{U}(1)$*-torsor morphism $\phi\colon T \to T$ is just a rotation.*

Exercise 13. *More generally, show that every G-torsor morphism $\phi\colon T \to T'$ is invertible, and show that the G-torsor automorphisms $\phi\colon T \to T$ form a group isomorphic to G. However, show that this group is not* canonically *isomorphic to G unless G is abelian!*

2 Classifying Spaces

Given a principal G-bundle $\pi\colon P \to M$ and a map $f\colon M' \to M$, we can 'pull back' π along f and get a principal G-bundle over M':

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d, "f^*\pi"'] & P \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Here

$$f^*P = \{(p,m') \in P \times M' \colon\ \pi(p) = f(m')\}$$

and $f^*\pi$ sends (p,m') to m'.

The marvelous thing is that *every* principal G-bundle over *every* space is isomorphic to the pullback of a single 'universal' one!

Theorem 14. *For any topological group G there is a* **classifying space** *BG equipped with a principal G-bundle*

$$\pi_G\colon EG \to BG$$

such that every principal G-bundle over every space M is isomorphic to f^*EG for some map $f\colon M \to BG$. We call f the **classifying map** *of the principal G-bundle f^*EG. Homotopic classifying maps give isomorphic principal G-bundles, and*

$$[M, BG] \cong \{\text{isomorphism classes of principal } G\text{−bundles over } M\}.$$

How can we construct this magical space BG? Naively, we might dream that BG should be the 'space of all G-torsors'. Then EG could be the principal G-bundle over BG whose fiber over any point T is just the torsor T itself! If we could make this precise, we could define the classifying map $f\colon M \to BG$ of any principal G-bundle P over M by

$$f(x) = P_x.$$

Check that then we would get $P = f^*EG$!

However, what topology should we put on the 'space of all G-torsors'? The discrete topology doesn't work, and it's hard to imagine any other. Furthermore, the 'space of all G-torsors' isn't even an honest set — it's a proper class!

To solve these problems we should instead consider the **category** of all G-torsors, and find an equivalent category that has a mere set of objects, instead of a proper class. Starting from this we can build BG in a way that makes our naive dream precise.

So, begin with the category GTor, for which:

- objects are G-torsors,
- morphisms are G-torsor morphisms.

Since all G-torsors are isomorphic to G, this category is equivalent to the category in which:

- the only object is G,
- the morphisms are G-torsor morphisms $\phi\colon G \to G$.

Note that such $\phi\colon G \to G$ are precisely left translations: if

$$\phi(gh) = \phi(g)h$$

for all $g, h \in G$, then

$$\phi(h) = \phi(1)h$$

for all $h \in G$. So, GTor is also equivalent to the category with:

- one object,
- elements of G as morphisms,

with multiplication in G as composition of morphisms. Equivalent categories are 'the same' for most purposes, so henceforth we will call this last category GTor. It has the advantage of being much smaller than the original version.

Moreover, this small version of GTor is a 'topological category':

Definition 15. *A* **topological category** *is a category where the set of all objects and the set of all morphisms are topological spaces, and all the category operations (source, target, composition, and the map sending any object to its identity morphism) are continuous.*

There's a standard way to turn a topological category into a topological space, and applying this to GTor gives BG! First, given a topological category C, we can form its **nerve** NC, which is a simplicial space:

Second, given any simplicial space X, we can form its **geometric realization** $|X|$, which is a space — just take the simplices literally and glue them together using face and degeneracy maps, but defining the topology with the help of the topology on each space of n-simplices. (You may be more familiar with the geometric realization of a simplicial *set*; this is like that but a little fancier.)

Composing these constructions, we define

$$BG = |N(G\mathrm{Tor})|.$$

So, BG looks like this:

Similarly, we build EG using the category of pointed G-torsors, $G\mathrm{Tor}_*$. Since all G-torsors are isomorphic to G, this is equivalent to the category in which:

- the objects are (G, g) for any $g \in G$
- the morphisms are G-torsor morphisms $\phi\colon (G, g) \to (G, g')$ with $\phi(g) = g'$.

Since G-torsor morphisms from G to itself are just left translations, this in turn is equivalent to the category in which:

- objects are elements $g \in G$,
- a morphism $h\colon g \to g'$ is an element $h \in G$ with $hg = g'$.

This 'small' version of $G\mathrm{Tor}_*$ is a topological category in an obvious way, so we can define

$$EG = |N(G\mathrm{Tor}_*)|$$

and use the forgetful functor

$$F\colon G\mathrm{Tor}_* \to G\mathrm{Tor}$$

to define

$$\pi_G = |N(F)|.$$

This gives our universal G-bundle

$$\pi_G \colon EG \to BG.$$

Check that sitting over each point of BG there's a G-torsor, just as in our naive dream!

This is beautiful mathematics, but it may seem too abstract for some, so let's learn to visualize points of BG. A point in BG is a point in the n-simplex:

$$0 \le t_1 \le \cdots \le t_n \le 1$$

together with an n-tuple of elements of G. So, it looks like this:

It's nice to think of this as a finite collection of 'particles' on the unit interval with 'charges' taking values in the group G. However, there are equivalence relations coming from face and degeneracy maps:

Since particles of any charge can be born and die at the endpoints, it's even better think of a point in BG as a collection of G-charged particles on S^1:

with a topology on BG that allows continuous paths in which these particles move around, collide or split, and be born or die at the north pole. When particles collide, their charges multiply. In particular, when a particle with charge g collides with its 'antiparticle' with charge g^{-1}, they become a particle of charge 1, which is equivalent to no particle at all. So, there are paths in BG in which particle/antiparticle pairs are annihilated — or for that matter, created! Charge is conserved except at the north pole.

When A is an *abelian* topological group, BA is again an abelian topological group, with multiplication defined like this:

(Note that we need A to be abelian for this product to be continuous.) So, in this case we can form BBA, $BBBA$, and so on. If you think about it a while, you'll see that a point in $B^n A$ is a finite collection of A-charged particles on S^n:

with a topology that allows continuous paths in which these particles move around, collide or split, and be born or die at the north pole. When particles with charges g and h collide, they form a particle of charge gh. Note that when $n \geq 2$, we need A to be abelian for this to make sense: there's no good way to say which particle is on the left and which particle is on the right!

Let us summarize the story so far. We have seen that to formalize the concept of 'parallel transport' we need the concept of a connection on a principal bundle. A principal G-bundle has G-torsors as fibers. Starting from the category of all G-torsors, we can build a space BG which has a 'universal' G-bundle on it: all others are pullbacks of this one.

In fact, the story goes much further. Whenever G is a compact Lie group, BG is a kind of infinite-dimensional manifold, and there's a nice way to put a connection on the universal G-bundle over BG. Given a smooth map from a manifold M to BG, we can pull back not only the universal bundle but also this connection to get a principal G-bundle *with connection* on M.

Even better, we can concoct certain closed differential forms on M from any principal G-bundle with connection over M. These represent elements of deRham cohomology called 'characteristic classes', which are independent of the choice of connection. When we apply this construction to the universal G-bundle over BG, these characteristic classes give us the whole real cohomology of BG. The characteristic classes of other principal G-bundles over other manifolds are simply the pullbacks of these.

Of course algebraic topologists are comfortable simply defining characteristic classes to be elements of the cohomology of BG, with coefficients in whatever you like. But if we restrict attention to the real cohomology, working with connections allows us to reason *geometrically* about characteristic classes. This has become very important in applications of topology to mathematical physics... and applications of mathematical physics to topology!

Here are some exercises to help you become more comfortable with the ideas of this lecture:

Exercise 16. *Use the particle picture of BBA to show that $BB\mathbb{Z} \cong \mathbb{C}P^\infty$, and to get an explicit abelian group structure on $\mathbb{C}P^\infty$. (Hint: think of integers* *labelling points of S^2 as orders of the zeroes or poles of a nonzero rational function on the Riemann sphere. This data determines the rational function up to a nonzero scalar multiple.*

Exercise 17. *Give a 'particle picture' of points in EG similar to that for BG. (Hint: the big difference is that now charge is conserved even at the north pole.) Use this to explicitly describe the map $\pi_G \colon EG \to BG$. Show that this is a principal G-bundle.*

Exercise 18. *Define a concept of 'equivalent' topological categories that generalizes the usual notion of equivalent categories. Show that if C and C' are equivalent topological categories, the spaces $|N(C)|$ and $|N(C')|$ are homotopy equivalent. Show that the topological category $G\mathrm{Tor}_*$ is equivalent to the topological category with just one object and one morphism. Conclude that $EG = |N(G\mathrm{Tor}_*)|$ is contractible.*

Exercise 19. *Show that for any topological group G, BG is connected.*

Exercise 20. *Using the previous two exercises, show that*

$$G \simeq \Omega BG$$

where Ω stands for based loop space of a pointed space, and BG becomes a pointed space using the fact that $N(G\mathrm{Tor})$ has just one 0-simplex. (Hint: you can use the long exact homotopy sequence of the fibration $G \to EG \to BG$.) Indeed, many authors define BG to be any connected pointed space with $\Omega BG \simeq G$.

3 A Taste of Higher Gauge Theory

We have seen that we can iterate the classifying space construction if we start with an abelian topological group. So, starting from any discrete abelian group A we can build a sequence of spaces

$$K(A,n) = B^n A$$

The space $K(A,0)$ is just A itself, and by Exercise 20 we have

$$\Omega K(A,n) = K(A,n-1),$$

so a little work shows that $K(A,n)$ is a space with A as its nth homotopy group, and with all its other homotopy groups being trivial. In fact this property characterizes the spaces $K(A,n)$ up to weak homotopy equivalence. We call $K(A,n)$ the nth **Eilenberg–Mac Lane space** of A.

Any decent homotopy theorist knows more about these spaces than I do, but I would like to say a bit about how they show up in 'higher gauge theory'. This is a generalization of gauge theory that deals with parallel transport, not of particles, but of strings or higher-dimensional membranes.

In this lecture I'll be a bit sketchy at times, because my main goal is to get you used to a wonderfully mind-boggling idea, and technical details would merely be distracting. I'll fill in more details next time.

To get the main idea across let's consider the example $A = \mathbb{Z}$. We have

$$\begin{array}{rcl} K(\mathbb{Z},0) & = & \mathbb{Z} \\ K(\mathbb{Z},1) & \simeq & \mathrm{U}(1) \\ K(\mathbb{Z},2) & \simeq & \mathbb{C}P^\infty. \end{array}$$

The first two are easy to see, and I gave you a nice way to see the third fact in Exercise 16. The space $K(\mathbb{Z},n)$ is a bit less familiar for $n > 2$, but we have seen that it consists of 'collections of $\mathbb{Z}$-charged particles on S^n'.

Now let's consider a puzzle about the meaning of cohomology groups. Any homotopy theorist worthy of the name knows that the nth cohomology group of a space X with coefficients in A classifies maps from X to $K(A,n)$:

$$H^n(X,A) = [X, K(A,n)].$$

But there is also *another* nice way to think about cohomology, at least when $A = \mathbb{Z}$. Since $K(\mathbb{Z},1) = \mathrm{U}(1)$ we have:

$$H^1(X,\mathbb{Z}) \cong \{\text{homotopy classes of } \mathrm{U}(1)\text{−valued functions on } X\}.$$

Since $K(\mathbb{Z},2) = B\mathrm{U}(1)$ is the classifying space for principal U(1)-bundles, we have:

$$H^2(X,\mathbb{Z}) \cong \{\text{isomorphism classes of principal } \mathrm{U}(1) \text{ bundles over } X\}.$$

There seems to be some pattern here. The puzzle is, *what comes next?*

Of course $K(\mathbb{Z},3) = BB\mathrm{U}(1) = B(\mathbb{C}P^\infty)$, so

$$H^3(X,\mathbb{Z}) \cong \{\text{isomorphism classes of principal } \mathbb{C}P^\infty \text{ bundles over } X\},$$

but this answer misses the point. We want a very 'U(1)-ish' description of the integral third cohomology of a space, which continues the pattern we've seen for H^1 and H^2.

The right answer is:

$$H^3(X,\mathbb{Z}) \cong \{\text{equivalence classes of } \mathrm{U}(1) \text{ gerbes over } X\},$$

or in my own preferred (but less standard) terminology,

$$H^3(X,\mathbb{Z}) \cong \{\text{equivalence classes of principal } \mathrm{U}(1)\ 2-\text{bundles over } X\}.$$

The idea here is simple but quite mind-boggling at first — which is why it's so fun. Just as a principal U(1)-bundle over X is a gadget where the fiber over each point looks like U(1), a principal U(1)-2-bundle is a gadget where the fiber over each point looks like U(1)Tor!

Of course U(1)Tor is not a topological space but a *topological category*, so the kind of gadget we need here is a generalization of a principal bundle which has topological categories rather than spaces as fibers. Replacing ordinary mathematical gadgets by analogous gadgets using categories in place of sets is called 'categorification'. So, what we need is to categorify the concept of bundle — hence the term '2-bundle'.

Let's see if we can make sense of this. When we say the fibers of a principal U(1)-bundle 'look like U(1)', we really mean that they are U(1)-torsors. So, by analogy, when we say that the fibers of a principal U(1)-2-bundle look like U(1)Tor, we must mean that they are U(1)Tor-torsors. But what in the world is a 'U(1)Tor-torsor'?

To figure this out, we need to realize that the category U(1)Tor is very much like an abelian group in its own right.

First of all, we can 'multiply' U(1)-torsors. Given U(1)-torsors T and T', we can define

$$T \otimes T' = \frac{T \times T'}{(tg,t') \sim (t,gt')} \,.$$

This definition should remind you of the tensor product of a right module of a ring with a left module of the same ring. Note that we are using the fact that U(1) is abelian here: this lets us make the right U(1)-space T' into a left U(1)-space by defining $gt' := t'g$. The space $T \otimes T'$ becomes a U(1)-torsor in an obvious way:

$$(t,t')g = (t,t'g).$$

Second of all, U(1) itself is the 'unit' for this multiplication: there are canonical isomorphisms

$$\begin{array}{rcl} \mathrm{U}(1) \otimes T & \cong & T \\ (g,t) & \mapsto & gt \end{array}$$

and

$$\begin{array}{rcl} T \otimes \mathrm{U}(1) & \cong & T \\ (t,g) & \mapsto & tg \end{array}$$

where again we use the fact that U(1) is abelian to make T into a left U(1)-space.

Third of all, every U(1)-torsor T has an 'inverse' T^{-1}. As a space, T^{-1} is just T, but we define tg in T^{-1} to be the element tg^{-1} in T. Again this works only because U(1) is abelian. We get canonical isomorphisms

$$T \otimes T^{-1} \cong \mathrm{U}(1),$$

$$T^{-1} \otimes T \cong \mathrm{U}(1).$$

So, the category U(1)Tor acts a lot like a group! Such a categorified version of a group is called a '2-group'. For more information on 2-groups, try my paper with Aaron Lauda [5]. Everything you can do with groups, you can do with 2-groups. For example, any pointed space has a fundamental 2-group. This is part of a massive pattern linking categorification to homotopy theory [4].

Since U(1)Tor acts a lot like a group, we can define torsors for it. Again I will be quite sketchy. Just as we can talk about right G-sets for a group G, we can talk about 'right G-categories' for a 2-group G. A right G-category X is a category equipped with a right action of G, meaning that there's a functor

$$X \times G \to X$$

satisfying the usual laws for a right action. We say a right G-category T is a 'G-torsor' if it is equivalent to G as a right G-category.

So, the concept of a U(1)Tor-torsor actually makes sense. And, we can go ahead and use this categorify the whole theory of principal U(1)-bundles and connections on these. To do this, we start by defining the concept of a 'principal U(1)-2-bundle over X'. You can find the details elsewhere [9, 10]; for now it is enough to imagine a gadget whose fibers over each point of X are not topological spaces but topological categories: in fact, U(1)Tor-torsors. And, the beautiful thing is that we obtain

$$H^3(X,\mathbb{Z}) \cong \{\text{equivalence classes of principal U(1) } 2 - \text{bundles over } X\}.$$

In fact, this sort of result was first proved not in the language of bundles but in the language of sheaves. A categorified sheaf is called a 'stack' [23]. A stack that's a categorified version of the sheaf of sections of a principal U(1) bundle is called a 'U(1) gerbe' [14, 16, 19, 24], and one has

$$H^3(X,\mathbb{Z}) \cong \{\text{equivalence classes of U(1) gerbes over } X\}.$$

I just happen to like bundles a bit more than sheaves, so I've been developing the 2-bundle formalism as a complement to the gerbe formalism. People have studied connections both on gerbes [12, 14, 26] and on 2-bundles [8]. The subject gets really interesting when one passes from 'abelian' 2-groups like U(1)Tor to more general 2-groups, since then the principal 2-bundles are classified by something called nonabelian cohomology, invented by Giraud [19]. A good place to read about this subject is Breen's book [11].

But the truly wonderful thing is that all this math shows up quite naturally in string theory [2, 3] and related mathematics like the study of Chern–Simons theory, central extensions of loop groups, and the 'string' group [7, 15]. The reason is that just as a connection on a principal bundle lets us talk about parallel transport of point particles, a connection on a principal 2-bundle lets us talk about parallel transport of curves — i.e., strings!

Here's an easy way to see why this might be true, at least in the abelian case. A smooth map from some manifold M to $B\mathrm{U}(1) \simeq K(\mathbb{Z}, 2)$:

$$f\colon M \to B\mathrm{U}(1)$$

gives a principal U(1) bundle with connection on M. But now suppose we have a smooth map from M to $BB\mathrm{U}(1) \simeq K(\mathbb{Z}, 3)$:

$$f\colon M \to BB\mathrm{U}(1).$$

This gives a principal U(1)-2-bundle with connection on M. But, we can 'loop' f to get a map

$$\Omega f\colon \Omega M \to B\mathrm{U}(1)$$

so we also get a principal U(1)-bundle with connection on ΩM. This gives a holonomy for any path in ΩM. But, a path in ΩM is a one-parameter family of loops in M! We can think of this as a 'closed string' tracing out a surface in M. So, we are getting a notion of parallel transport for strings.

There's much more to say about this, but alas, there's no time! I hope you look at some of the references for more of the story.

For now, all I can do is to give a hint as to why principal U(1)-2-bundles should be classified by third integral cohomology, just as principal U(1)-bundles are classified by second cohomology. The reason is that $K(\mathbb{Z}, 3)$ is related to U(1)Tor just as $K(\mathbb{Z}, 2)$ is related to U(1). In fact, just as

$$K(\mathbb{Z}, 2) \simeq B\mathrm{U}(1),$$

we have

$$K(\mathbb{Z}, 3) \simeq B(\mathrm{U}(1)\mathrm{Tor}).$$

To really make sense of this this we'd need to understand what we mean by the *classifying space of an abelian topological 2-group.* I'll tackle this issue in the next lecture. In fact, next time I'll show that the pattern continues indefinitely:

$$H^4(X, \mathbb{Z}) \cong \{\text{equivalence classes of principal U(1)3} - \text{bundles over } X\}.$$

because

$$K(\mathbb{Z}, 4) \simeq B((\mathrm{U}(1)\mathrm{Tor})\mathrm{Tor})$$

and so on.

But to prepare ourselves for this climb, let's just try to imagine how to build $B(\mathrm{U}(1)\mathrm{Tor})$ — and why it might turn out to be $K(\mathbb{Z}, 3)$. To do this, let's recall how we built $B\mathrm{U}(1)$, and categorify that.

In the previous lecture we built $B\mathrm{U}(1)$ by taking U(1)Tor and turning it into a space. By analogy, we must build $B(\mathrm{U}(1)\mathrm{Tor})$ by forming something called (U(1)Tor)Tor and turning *that* into a space! This actually makes some sense: we've seen that U(1)Tor is a categorified version of a group, so we can define torsors for it. These torsors form a gadget called (U(1)Tor)Tor. But what sort of gadget is it, and how do we turn it into a space?

In fact, this gadget is a topological '2-category'. A 2-category is a categorified version of a category: instead of a mere *set* $\hom(x, y)$ of morphisms from any object x to any object y, this has a *category* $\hom(x, y)$. We draw the objects of $\hom(x, y)$ as arrows:

$$f\colon x \to y$$

and draw the morphisms of $\hom(x, y)$ as 'arrows between arrows':

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

or

$$\alpha\colon f \Rightarrow g$$

for short. We call the arrows 'morphisms' in our 2-category, and call the arrows between arrows '2-morphisms'.

If the notion of 2-category makes your head spin, don't worry — enjoy the sensation while it lasts! Some people pay good money to get dizzy by riding roller-coasters in amusement parks. We mathematicians are lucky enough to get *paid* for making ourselves dizzy with abstract concepts. After a while 2-categories become as routine as second derivatives or double integrals, and then you have to move on to more abstract structures to get that thrill.

The 2-category you're most likely to have seen is Cat. This has:

- categories as objects,
- functors as morphisms,
- natural transformations as 2-morphisms.

Indeed, (U(1)Tor)Tor is closely related to Cat, because U(1)Tor-torsors are certain categories with extra structure. Roughly speaking, (U(1)Tor)Tor has:

- U(1)Tor-torsors as objects,
- functors preserving the right U(1)Tor action as morphisms,
- natural transformations as 2-morphisms.

But in fact, just as all U(1)-torsors are isomorphic, all U(1)Tor-torsors are equivalent. This lets us find a nice small 2-category equivalent to (U(1)Tor)Tor, just as we found a nice small category equivalent to U(1)Tor in the last lecture. Last time we saw that U(1)Tor was equivalent to the category with:

- one object,
- elements of U(1) as morphisms.

Similarly, it turns out that (U(1)Tor)Tor is equivalent to the 2-category with:

- one object,
- one morphism,
- elements of U(1) as 2-morphisms.

And, it's easy to make this into a *topological* 2-category, using the topology on U(1).

In short, (U(1)Tor)Tor is just like U(1)Tor 'shifted up a notch'. So, it should not be surprising that once we learn how to turn a topological 2-category into a space, (U(1)Tor)Tor gives the space $K(\mathbb{Z}, 3)$, just as U(1)Tor gave $K(\mathbb{Z}, 2)$. Nor should it be surprising that this pattern continues on into higher dimensions!

4 Higher Torsors and Eilenberg–Mac Lane Spaces

Now let's make some of the ideas of the previous lecture more precise. We'll build a recursive machine that lets us define n-categories for all n, and construct a topological n-category of 'n-torsors' for any abelian topological group A. The geometric realization of the nerve of this will be the iterated classifying space $B^n A$. So, for example, we get

$$|N(\mathrm{U}(1)n\mathrm{Tor})| = K(\mathbb{Z}, n+1).$$

First we need the notion of an 'enriched category'. To get ahold of this, first look at this definition of 'category':

> A **category** C has a class of objects, and for each pair of objects x, y a <u>set</u> $\hom(x, y)$, and for each triple of objects a composition <u>function</u>
>
> $$\circ\colon \hom(x,y) \times \hom(y,z) \to \hom(x,z)$$
>
> such that....

Note that the category of sets plays a special role in the underlined terms. If we replace this category by some other category K, we get the concept of a **category enriched over** K, or K**-category** for short. We assume that K has finite products so that the '$\times$' in the composition function makes sense. Here's how it goes:

> A K**-category** C has a class of objects, and for each pair of objects x, y an <u>object of K</u> $\hom(x, y)$, and for each triple of objects a composition <u>morphism</u>
>
> $$\circ\colon \hom(x,y) \times \hom(y,z) \to \hom(x,z)$$
>
> such that....

We leave it as an exercise (see below) to fill in the the details.

Enriched categories are not really strange: nature abounds with them. For example, the category Top is enriched over itself: for any pair of spaces x, y there is a space $\hom(x, y)$ of contiuous maps from x to y, and composition is a continous map:

$$\circ\colon \hom(x,y) \times \hom(y,z) \to \hom(x,z).$$

The category of abelian groups is also enriched over itself.

We can define K-functors between K-categories: given K-categories C and C', a K-functor $F\colon C \to C'$ consists of a map sending objects of C to objects of C', and also for any pair of objects $x, y \in C$ a morphism

$$F\colon \hom(x,y) \to \hom(Fx, Fy).$$

As usual, we require that identities and composition are preserved.

In fact, there is a category $K\mathrm{Cat}$ in which:

- objects are K-categories,
- morphisms are K-functors.

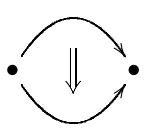
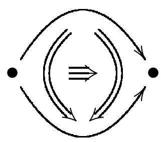
Even better, $K\mathrm{Cat}$ has finite products! This allows us to iterate this construction and define n-categories in a recursive way:

Definition 21. *Define the category* $0\mathrm{Cat}$ *to be the category of sets, and for* $n \geq 0$ *define*

$$(n+1)\mathrm{Cat} = (n\mathrm{Cat})\mathrm{Cat}.$$

An object of $n\mathrm{Cat}$ *is called an* n**-category***, and a morphism of* $n\mathrm{Cat}$ *is called an* n**-functor**.

If one unravels this above definition, one sees that a (strict) n-category consists of objects, morphisms between objects, 2-morphisms between morphisms, and so on up to n-morphisms. The j-morphisms are shaped like j-dimensional 'globes':

Objects	**Morphisms**	**2-morphisms**	**3-morphisms**	j-morphisms
$\bullet$	$\bullet \longrightarrow \bullet$			j-dimensional globes

There are various geometrically sensible ways to compose j-morphisms, satisfying various 'associativity' and 'identity' laws. All this can be extracted from the slick definition we have given above!

To be honest, we have only defined so-called 'strict' n-categories, where all the laws hold 'on the nose', as equations. We won't be needing the vastly more interesting and complicated 'weak' n-categories, even though these are more important for topology [4]. The reason we can get away with this is that we're only talking about the simplest case of higher gauge theory, where the classifying space is an Eilenberg–Mac Lane space. The lack of interesting Postnikov data in these spaces lets us build them from strict n-categories.

Next let us define an n-category $A\,n\mathrm{Tor}$ whose objects are n-torsors for the abelian group A. First note that we can define the concept of 'group' in any category with finite products. To do this, we just take the usual definition of a group, write it out using commutative diagrams, and let these diagrams live in K. Skeptics may want to see the details:

Definition 22. *Given a category K with finite products, a* **group in** K *is*

- *an object G of K,*

together with

- *a multiplication morphism $m\colon G \times G \to G$,*
- *an identity for the multiplication, given by the morphism $\mathrm{id}\colon I \to G$ where I is the terminal object in K,*
- *a 'inversion' morphism $\mathrm{inv}\colon G \to G$,*

such that the following diagrams commute:

- *the associative law:*

$$\begin{tikzcd} & G \times G \times G \arrow[dl, "m\times 1"'] \arrow[dr, "1\times m"] & \\ G \times G \arrow[dr, "m"'] & & G \times G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

- *the right and left unit laws:*

$$\begin{tikzcd} I \times G \arrow[r, "\mathrm{id}\times 1"] \arrow[dr] & G \times G \arrow[d, "m"] & G \times I \arrow[l, "1\times \mathrm{id}"'] \arrow[dl] \\ & G & \end{tikzcd}$$

- *the right and left inverse laws:*

$$
\begin{tikzcd}
& G \times G \arrow[r, "1\times\mathrm{inv}"] & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "\Delta"] \arrow[drr] & & & G \\
& & I \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& G \times G \arrow[r, "\mathrm{inv}\times 1"] & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "\Delta"] \arrow[drr] & & & G \\
& & I \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
$$

where $\Delta\colon G \to G \times G$ is the diagonal map.

Example 23. *A group in the category of topological spaces is a topological group; a group in the category of smooth manifolds is a Lie group.*

Now suppose G is a group in K. Copying our work in Section 2, we define a K-category called GTor with

- one object, say $*$.
- $\hom(*, *) = G$,

with the multiplication in G as composition of morphisms. This is equivalent to the GTor we know and love when K is the category of sets.

We can easily define the concept of an 'abelian' group in K, and this is when we can iterate the torsor construction. Suppose A is an abelian group in K. Then ATor is an object in KCat, but in fact it's even better: it is an abelian group in KCat! So, we are ready to iterate:

$$A \text{ is an abelian group in } K \Longrightarrow$$

$$A\mathrm{Tor} \text{ is an abelian group in } K\mathrm{Cat}.$$

Definition 24. *Suppose A is an abelian group. Define $A0$Tor to be A, and for $n \geq 0$ define*
$$A\,(n+1)\mathrm{Tor} = (A\,n\mathrm{Tor})\mathrm{Tor}.$$
*AnTor is an n-category whose objects are called **A-n-torsors**.*

Note that $A\,n$Tor is an abelian group in nCat. In fact, if if A is a topological abelian group one can check that $A\,n$Tor is an abelian group in the category of 'topological n-categories'. Of course, one first needs to define the concept of a topological n-category, but this is not very hard. To keep things simple, however, we will focus on the case where A is a discrete abelian group, for example $\mathbb{Z}$.

Now that we have $A\,n$Tor in hand, we want to turn it into a space and show that this space is $K(A, n)$. To do this, we use two theorems:

Theorem 25 (Brown–Higgins) . *The category of abelian groups in nCat is equivalent to the category of degree-n chain complexes of abelian groups: that is, chain complexes of the form:*

$$A_0 \leftarrow A_1 \leftarrow \cdots \leftarrow A_n.$$

Theorem 26 (Dold–Kan) . *The category of degree-n chain complexes is equivalent to the category of simplicial abelian groups such that all j-simplices with $j > n$ are degenerate.*

The first theorem was probably known to Grothendieck quite a while back, but the first proof can be found in a paper by Brown and Higgins [13]. The second theorem is very famous, and it can be found in many textbooks [1]. The proofs are very similar: for each j, we take the j-dimensional simplices or globes in our simplicial abelian group or n-categorical abelian group and construct j-chains from them, defining the differential in a geometrically obvious way.

Now, suppose that A is an abelian group. Then $A\,n\mathrm{Tor}$ is an abelian group in nCat, and it has

- one object,
- one morphism,
- ......,
- one n-morphism for each element of A.

If we use the Brown–Higgins theorem this becomes a degree-n chain complex. Unsurprisingly, it looks like this:

$$0 \leftarrow 0 \leftarrow \cdots \leftarrow A.$$

We can then apply the Dold–Kan theorem and turn this into a simplicial abelian group. We can then take the geometric realization of this and get an abelian topological group Using the proof of the Dold–Kan theorem one can see that this topological group has

- $\pi_0 = 1$,
- $\pi_1 = 1$,
- ......,
- $\pi_n = A$

with all higher homotopy groups vanishing. So, it is $K(A,n)$! More generally, if we had started with a topological abelian group A, a version of this game would give us the n-fold classifying space B^nA.

I hope you see by now that we are really just talking about the same thing in many different guises here. Indeed, you might get the impression that the whole business is an elaborate shell game. But it's not: the relation of $B^n\mathrm{U}(1)$ to $\mathrm{U}(1)$-n-torsors allows us to think of integral cohomological classes on a space X as principal $\mathrm{U}(1)$-n-bundles on X, and deal with these geometrically, especially when X is a smooth manifold.

Furthermore, we should be able to generalize a lot of this story to principal n-bundles for nonabelian n-groups — that is, nonabelian groups in $(n-1)$Cat. So far this has only been demonstrated for $n = 2$ [8, 12], or higher n for the abelian case [17, 18, 26], but the results are already very interesting.

Unfortunately I have only had time to skim the surface. I didn't even get around to defining a 2-bundle! Alissa Crans, Danny Stevenson and I went further in our lectures at Ross Street's 60th birthday conference [9]. But I hope this tiny taste of higher gauge theory leaves you hungry for more.

Finally, here are few more exercises to try:

Exercise 27. *Suppose K is a category with finite products. Write down the complete definition of a K-category. (Hint: one needs to use the terminal object in K to generalize the clause saying that every object has an identity morphism.*

Instead of saying there is an element $1_x \in \hom(x,x)$ with certain properties, one must say there is a morphisms $i_x \colon I \to \hom(x,x)$ with certain properties, where I is the terminal object of K.)

Exercise 28. *Write down the complete definition of a K-functor, and show that $K\mathrm{Cat}$ is a category.*

Exercise 29. *Show that $K\mathrm{Cat}$ has finite products.*

If you get stuck on the above exercises, take a look at Max Kelly's book on enriched categories [21], which is now freely available online.

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