Higher Gauge Theory and the String Group

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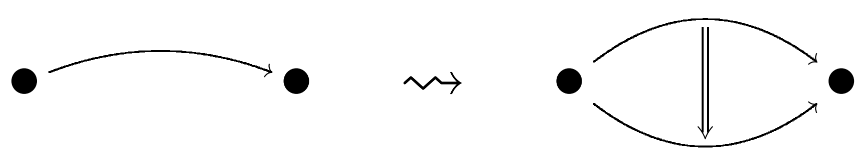
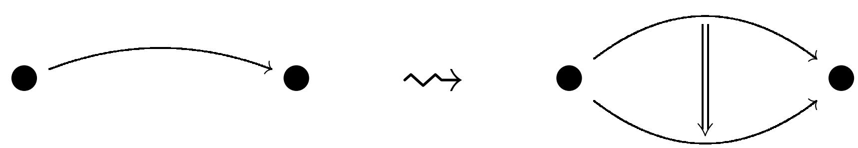


For more see: http://math.ucr.edu/home/baez/esi/

Categorification

$$\begin{array}{rcl}\textbf{sets} & \rightsquigarrow & \textbf{categories}\\ \textbf{functions} & \rightsquigarrow & \textbf{functors}\\ \textbf{equations} & \rightsquigarrow & \textbf{natural isomorphisms}\end{array}$$

Categorification 'boosts the dimension' by one:

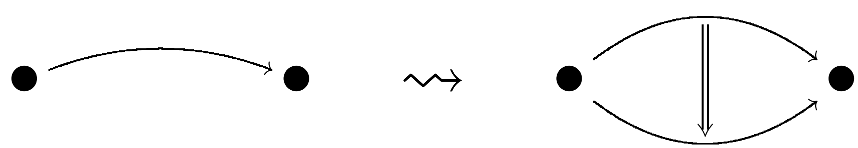


In **strict** categorification we keep equations as equations. This is evil... but today we'll do it whenever it doesn't cause trouble, just to save time.

Higher Gauge Theory

$$\begin{aligned}
\textbf{groups} &\rightsquigarrow \textbf{2-groups}\\
\textbf{Lie algebras} &\rightsquigarrow \textbf{Lie 2-algebras}\\
\textbf{bundles} &\rightsquigarrow \textbf{2-bundles}\\
\textbf{connections} &\rightsquigarrow \textbf{2-connections}
\end{aligned}$$

Connections describe parallel transport for particles.
2-Connections describe parallel transport for strings!



We should even go beyond $n = 2$... but not today.

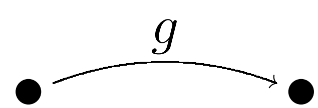
Fix a simply-connected compact simple Lie group G. Then:

- The Lie algebra $\mathfrak{g}$ gives a 1-parameter family of Lie 2-algebras $\mathfrak{string}_k(\mathfrak{g})$.
- When $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ comes from a Lie 2-group $\mathrm{String}_k(G)$.
- The 'geometric realization of the nerve' of $\mathrm{String}_k(G)$ is a topological group, $|\mathrm{String}_k(G)|$.
- Principal $\mathrm{String}_k(G)$-2-bundles are the same as $|\mathrm{String}_k(G)|$-bundles.
- For $k = 1$, $|\mathrm{String}_k(G)|$ is G with its 3rd homotopy group made trivial.
- We can define connections and characteristic classes for $\mathrm{String}_k(G)$-2-bundles!

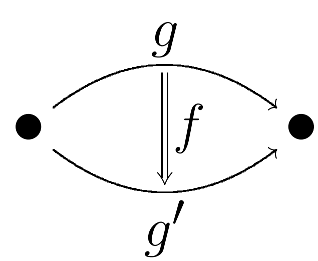
2-Groups

A **strict 2-group** is a category in Grp: a category with a group of objects and a group of morphisms, such that all the category operations are group homomorphisms.

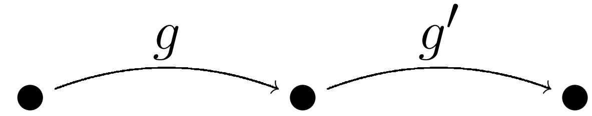
We draw the objects in a 2-group like this:



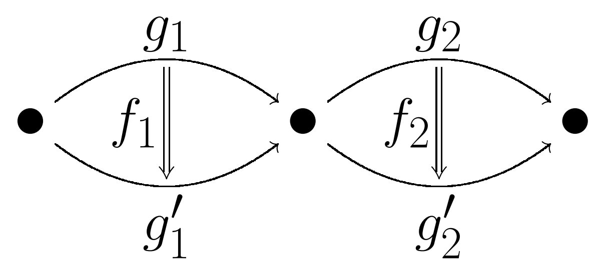
We draw the morphisms like this:



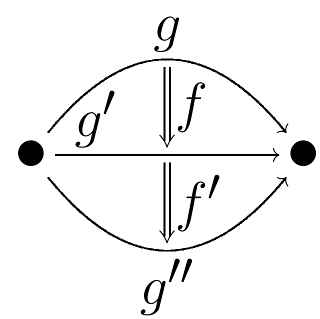
We can multiply objects:



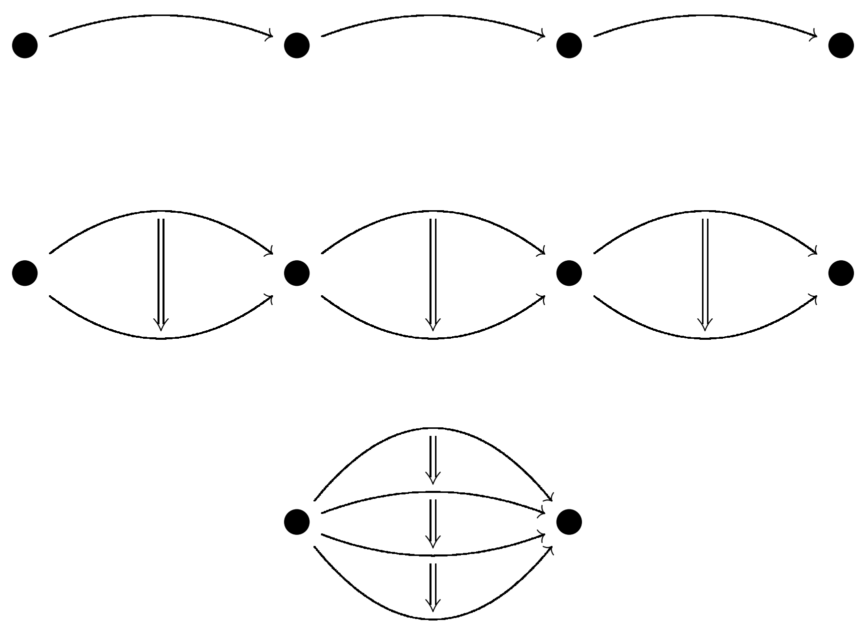
multiply morphisms:



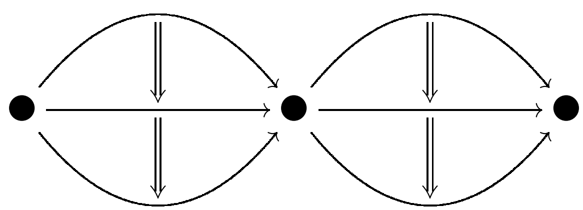
and compose morphisms:



All 3 operations have a unit and inverses. All 3 are associative, so these are well-defined:



Finally, the **interchange law** holds, meaning



is well-defined.

Lie 2-Algebras

A **strict Lie 2-algebra** is a category in LieAlg: a category with a Lie algebra of objects and a Lie algebra of morphisms, such that all the category operations are Lie algebra homomorphisms.

The theory of strict Lie 2-algebras closely mimics the theory of 2-groups. For example...

Theorem (Mac Lane, Sinh). A 2-group is determined up to equivalence by:

- the group G of isomorphism classes of objects,
- the abelian group A of endomorphisms of any object,
- an action of G on A,
- an element of $H^3(G, A)$.

Theorem (Gerstenhaber, Crans). A Lie 2-algebra is determined up to equivalence by:

- the Lie algebra $\mathfrak{g}$ of isomorphism classes of objects,
- the vector space $\mathfrak{a}$ of endomorphisms of any object,
- a representation of $\mathfrak{g}$ on $\mathfrak{a}$,
- an element of $H^3(\mathfrak{g}, \mathfrak{a})$.

Suppose G is a simply-connected compact simple Lie group. Let $\mathfrak{g}$ be its Lie algebra. A lemma of Whitehead says:

$$H^3(\mathfrak{g}, \mathbb{R}) = \mathbb{R}$$

So:

Corollary. For any $k \in \mathbb{R}$ there is a Lie 2-algebra $\mathfrak{string}_k(\mathfrak{g})$ for which:

- $\mathfrak{g}$ is the Lie algebra of isomorphism classes of objects;
- $\mathbb{R}$ is the vector space of endomorphisms of any object.

Every Lie 2-algebra with these properties is equivalent to $\mathfrak{string}_k(\mathfrak{g})$ for some unique $k \in \mathbb{R}$.

Theorem. For any $k \in \mathbb{Z}$, $\mathfrak{string}_k(\mathfrak{g})$ is the Lie 2-algebra of an infinite-dimensional Lie 2-group $\mathrm{String}_k(G)$.

Theorem. The morphisms in $\mathrm{String}_k(G)$ starting at the constant path form the level-k central extension of the loop group ΩG:

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \longrightarrow \Omega G \longrightarrow 1$$

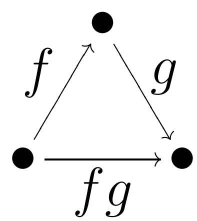
For any category $\mathcal{C}$ there is a space $|\mathcal{C}|$, the **geometric realization of the nerve** of $\mathcal{C}$, built from a vertex for each object:

$$\bullet\ x$$

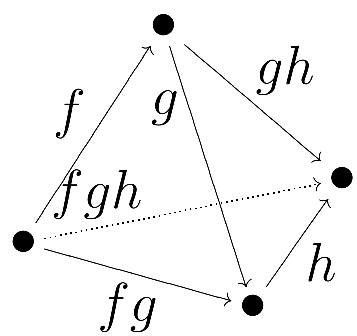
an edge for each morphism:

$$\bullet \xrightarrow{f} \bullet$$

a triangle for each composable pair of morphisms:



a tetrahedron for each composable triple:



and so on...

A 2-group is a category *with a product and inverses.* So, if $\mathcal{G}$ is a 2-group, $|\mathcal{G}|$ is a topological group.

More generally, we can define a topological group $|\mathcal{G}|$ for any *topological* 2-group $\mathcal{G}$.

Theorem. For any $k \in \mathbb{Z}$, there is a short exact sequence of topological groups:

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow |\mathrm{String}_k(G)| \xrightarrow{p} G \longrightarrow 1$$

where p is a fibration. When $k = 1$, this exhibits $|\mathrm{String}_k(G)|$ as the '3-connected cover' of G: the topological group formed by making the 3rd homotopy group of G trivial.

For example, start with $\mathrm{O}(n)$:

- Making π_0 trivial gives $\mathrm{SO}(n)$.
- Making π_1 trivial gives $\mathrm{Spin}(n)$.
- Making π_2 trivial still gives $\mathrm{Spin}(n)$.
- Making π_3 trivial gives the 3-connected cover... something new and interesting: the **string group**.

So, we're getting the string group from a 2-group.

2-Bundles — Quick and Dirty

For any topological 2-group $\mathcal{G}$ and any space X, we can define a **principal $\mathcal{G}$-2-bundle over** X to consist of:

- an open cover U_i of X,
- continuous maps

  $$g_{ij} \colon U_i \cap U_j \to \mathrm{Ob}(\mathcal{G})$$

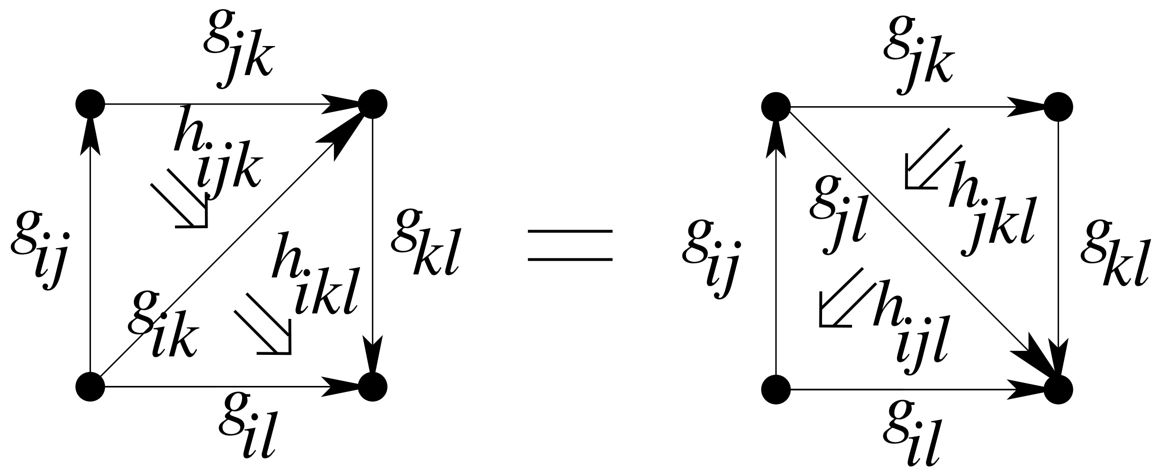
  satisfying $g_{ii} = 1$, and
- continuous maps

  $$h_{ijk} \colon U_i \cap U_j \cap U_k \to \mathrm{Mor}(\mathcal{G})$$

  with

  $$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \to g_{ik}(x)$$

satisfying the **nonabelian 2-cocycle condition:**



on any quadruple intersection $U_i \cap U_j \cap U_k \cap U_\ell$.

There's a natural notion of 'equivalence' for 2-bundles over X, since they form a 2-category.

Theorem. For any topological 2-group $\mathcal{G}$ and paracompact Hausdorff space X, there is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over X,
- isomorphism classes of principal $|\mathcal{G}|$-bundles over X,
- homotopy classes of maps $f\colon X \to B|\mathcal{G}|$.

So, $B|\mathcal{G}|$ is the classifying space for $\mathcal{G}$-2-bundles.

Characteristic Classes

Let G be a simply-connected compact simple Lie group, and let $\mathcal{G} = \mathrm{String}_k(G)$ with $k = 1$.

The homomorphism

$$|\mathcal{G}| \xrightarrow{p} G$$

gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|\mathcal{G}|, \mathbb{R})$$

This is onto, with kernel generated by the 'second Chern class' $c_2 \in H^4(BG, \mathbb{R})$.

In this case, the real characteristic classes of $\mathcal{G}$-2-bundles are just like those of G-bundles, but with the second Chern class killed!

There's a concept of 'connection' and 'curvature' for 2-bundles when $\mathcal{G}$ is a Lie 2-group.

In this case, we should be able to compute the real characteristic classes of a 2-bundle starting from a connection on this 2-bundle.

Sati, Schreiber and Stasheff have already made progress towards this.