Classical Mechanics Homework

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1 The 2-Body Problem

The goal of this problem is to understand a pair of particles interacting via a central force such as gravity. We'll reduce it to problem you've already studied — the case of a *single* particle in a central force.

Suppose we have a system of two particles interacting by a central force. Their positions are functions of time, say $q_1, q_2 \colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m_1 \ddot{q}_1 = f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|}$$

$$m_2 \ddot{q}_2 = f(|q_2 - q_1|) \frac{q_2 - q_1}{|q_2 - q_1|}.$$

Here m_1, m_2 are their masses, and the force is described by some smooth function $f \colon (0, \infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of momentum and symmetry under translations and Galilei boosts we can work in coordinates where

$$m_1 q_1(t) + m_2 q_2(t) = 0 \tag{1}$$

for all times t. This coordinate system is called the **center-of-mass frame**.

We could use equation (1) to solve for q_2 in terms of q_1, or vice versa, but we can also use it to express both q_1 and q_2 in terms of the **relative position**

$$q(t) = q_1(t) - q_2(t).$$

This is more symmetrical, so this is what we will do. Henceforth we only need to talk about q. Thus we have reduced the problem to a 1-body problem!

1. Now by taking the second time derivative of the relative postion and using equation (1) to solve for $\dot{q}_1$ in terms of $\dot{q}_2$, we have

$$\begin{aligned}
\ddot{q} &= \ddot{q}_1 - \ddot{q}_2 \\
&= \left(1 + \frac{m_1}{m_2}\right) \ddot{q}_1 \\
&= \left(\frac{m_1 + m_2}{m_1 m_2}\right) m_1 \ddot{q}_1 \\
&= \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(|q_1 - q_2|) \frac{q_1 - q_2}{|q_1 - q_2|}
\end{aligned}$$

where the last equality follows from Newton's 2^{nd} law. But note that this says:

$$m \ddot{q} = f(|q|) \frac{q}{|q|}$$

where m is the so-called **reduced mass**

$$m = \frac{m_1 m_2}{m_1 + m_2},$$

and this looks exactly like Newton's second law for a single particle!

2. We also have a similar single particle property involving the total energy. Indeed, by squaring both sides of equation (1) then adding $m_1 m_2(\dot{q}_1^2 - \dot{q}_2^2)$ to both sides, we have

$$m_1^2\dot{q}_1^2 + m_2^2\dot{q}_2^2 + 2m_1m_2\dot{q}_1\dot{q}_2 + m_1m_2(\dot{q}_1^2 - \dot{q}_2^2) = m_1m_2(\dot{q}_1^2 - \dot{q}_2^2),$$

or equivalently,

$$m_1(m_1 + m_2)\dot{q}_1^2 + m_2(m_1 + m_2)\dot{q}_2^2 = m_1m_2(\dot{q}_1 - \dot{q}_2)^2.$$

Now note the left side is $2(m_1 + m_2)$ times the total kinetic energy, thus we have a new way to write this total kinetic energy in terms of the reduced mass and relative postion, that is

$$\frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 = \frac{1}{2}m|\dot{q}|^2.$$

Therefore, the total energy can be written as

$$E = \frac{1}{2}m\,|\dot{q}|^2 + V(|q|)$$

and this looks exactly like the energy of a single particle!

3. It should be no suprise that we'll obtain a similar suprise for the total angular momentum J of these two bodies. Indeed,

$$\begin{aligned}
J &= m_1q_1 \times \dot{q}_1 + m_2q_2 \times \dot{q}_2 \\
&= m\left(1 + \frac{m_1}{m_2}\right)q_1 \times \dot{q}_1 + m\left(1 + \frac{m_2}{m_1}\right)q_1 \times \dot{q}_2 \\
&= mq_1 \times \dot{q}_1 + q_1 \times \frac{m_1}{m_2}\dot{q}_1 + mq_2 \times \dot{q}_2 + q_2 \times \frac{m_2}{m_1}\dot{q}_2 \\
&= mq_1 \times (\dot{q}_1 - \dot{q}_2) - mq_2 \times (\dot{q}_1 - \dot{q}_2) \\
&= mq \times \dot{q}
\end{aligned}$$

and this looks exactly like the angular momentum of a single particle!

At this point we're back to a problem we've already solved: a *single* particle in a central force. The only difference is that now q stands for the *relative* position and m stands for the *reduced* mass!

So, we instantly conclude that two bodies orbiting each other due to the force of gravity will *both* have an orbit that's either an ellipse, or a parabola, or a hyperbola... when viewed in the center-of-mass frame.

2 Poisson brackets

Let $\mathbb{R}^{2n}$ be the **phase space** of a particle in $\mathbb{R}^n$, with coordinates q_i, p_i $(1 \leq i \leq n)$. Let $C^\infty(\mathbb{R}^{2n})$ be the set of smooth real-valued functions on $\mathbb{R}^{2n}$, which becomes an commutative algebra using pointwise addition and multiplication of functions.

We define the **Poisson bracket** of functions $F, G \in C^\infty(\mathbb{R}^{2n})$ by:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

4. The Poisson brackets make the vector space $C^\infty(R^{2n})$ into a **Lie algebra**.
Proof: Antisymmetry of the bracket is immediate from the definition and bilinearity follows from the linearity of the differential operator. However, some work is required to show the Jacobi identity. We first assume that in which we are asked to prove in question 5, namely that the Poisson bracket and the multiplication in the commutative algebra $C^\infty(\mathbb{R}^{2n})$ are related through the Leibniz identity. Let $A, B, C \in C^\infty(\mathbb{R}^{2n})$, for completeness we note:

$$\frac{\partial}{\partial x}\{A, B\} = \sum_{i=1}^{n} \left(\frac{\partial^2 A}{\partial x \partial p_i}\frac{\partial B}{\partial q_i} + \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial x \partial p_i}\right) - \left(\frac{\partial^2 A}{\partial x \partial q_i}\frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial x \partial q_i}\right)$$

and

$$\{\{A, B\}, C\} = \sum_{i=1}^{n} \frac{\partial\{A, B\}}{\partial p_i}\frac{\partial C}{\partial q_i} - \frac{\partial\{A, B\}}{\partial q_i}\frac{\partial C}{\partial q_i}.$$

We now see that a permutation in the letters G and H in the expression $\{\{F, G\}, H\}$ fixes all terms that contain a factor of F under a second order differential operator. Therefore, in the expression $\{\{F, G\}, H\} - \{\{F, H\}, G\}$, these terms are killed and thus can be written in the following form:

$$\sum_{i=1}^{n} \left(\left\{F, \frac{\partial G}{\partial p_i}\right\}\frac{\partial H}{\partial q_i} - \left\{F, \frac{\partial G}{\partial q_i}\right\}\frac{\partial H}{\partial p_i}\right) - \left(\left\{F, \frac{\partial H}{\partial p_i}\right\}\frac{\partial G}{\partial q_i} - \left\{F, \frac{\partial H}{\partial q_i}\right\}\frac{\partial G}{\partial p_i}\right).$$

However, the Leibinz identity says this can be reduced further to

$$\sum_{i=1}^{n} \left(\left\{F, \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i}\right\} - \left\{F, \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i}\right\}\right).$$

Now anitsymmetry and bilinearity give us the following:

$$\begin{aligned}
\{\{F, G\}, H\} + \{G, \{F, H\}\} &= \{\{F, G\}, H\} - \{\{F, H\}, G\} \\
&= \left\{F, \sum_{i=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i}\right\} \\
&= \{F, \{G, H\}\}
\end{aligned}$$

and this is the Jacobi identity. To complete the proof we are left to independently verify the Leibniz identity. But since no one is actually reading this, it will suffice to sit in my chair and shout out loud the word refrigerator as I type it.

(*Note the Jacobi identity resembles the product rule $d(GH) = (dG)H + GdH$, with bracketing by F playing the role of d. This is no accident!*)

5. The Poisson brackets and ordinary multiplication of functions make the vector space $C^\infty(\mathbb{R}^{2n})$ into a **Poisson algebra**.
Proof: We need only to verify the Leibniz identity:

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$

Indeed,

$$\begin{aligned}
\{F, GH\} &= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial GH}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial GH}{\partial p_i} \\
&= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial q_i}H + G\frac{\partial H}{\partial q_i}\right) - \frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial p_i}H + G\frac{\partial H}{\partial p_i}\right) \\
&= \left(\sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right)H + G\left(\sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i}\right) \\
&= \{F, G\}H + G\{F, H\}.
\end{aligned}$$

(*Again this identity resembles the product rule!*)