Classical Mechanics Homework
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Angular Momentum and Rotations

In this problem we will see that angular momentum generates rotations for a particle in $\mathbb{R}^n$. We begin by recalling a bit about rotations. Let $\mathrm{O}(n)$ be the **orthogonal group**: the group of all linear transformations of $\mathbb{R}^n$ that preserve distances. We can describe an element $R \in \mathrm{O}(n)$ as a real $n \times n$ matrix that is **orthogonal**, meaning

$$RR^* = R^*R = I$$

where R^* is the adjoint of the matrix R and I is the identity matrix.

We can define the **exponential** of any $n \times n$ real matrix A to be the matrix defined by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

(This series always converges.) Some easy calculations show that

$$\exp((s+t)A) = \exp(sA)\exp(tA)$$

for all $s, t \in \mathbb{R}$. Also, the entries of the matrix $\exp(tA)$ are smooth functions of $t \in \mathbb{R}$.

1. Suppose that A is **skew-adjoint**, meaning $A^* = -A$. Show that $\exp(tA) \in \mathrm{O}(n)$ for all $t \in \mathbb{R}$.

The group $\mathrm{O}(n)$ includes both rotations and reflections. In particular, $\mathrm{O}(n)$ consists of two connected components — the component where $\det(R) = 1$ and the component where $\det(R) = -1$. We define the **rotation group** or **special orthogonal group** $\mathrm{SO}(n)$ to be the subgroup consisting of all $R \in \mathrm{O}(n)$ with $\det(R) = 1$. This subgroup only includes rotations. A continuous curve can never go from one component to another. So, if A is skew-adjoint, $\exp(tA)$ must actually lie in $\mathrm{SO}(n)$ for all t.

We define $\mathfrak{so}(n)$ to be the set of all skew-adjoint real $n \times n$ matrices. This set $\mathfrak{so}(n)$ is actually a Lie algebra, since it is a vector space closed under the bracket operation $[x, y] = xy - yx$. It is called the **Lie algebra of the rotation group**.

Now, let $\mathbb{R}^{2n}$ be the phase space for a particle in $\mathbb{R}^n$. A point $(q, p) \in \mathbb{R}^{2n}$ describes the particle's **position** $q \in \mathbb{R}^n$ and **momentum** $p \in \mathbb{R}^n$. The algebra of smooth real-valued functions $C^\infty(\mathbb{R}^{2n})$ becomes a Poisson algebra with

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i}.$$

2. Given $A \in \mathfrak{so}(n)$, let

$$\phi \colon \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

be given by

$$\phi(t, q, p) = (\exp(tA)q, \exp(tA)p).$$

Using the facts I've told you, show that ϕ is a flow.

(For example, in 3 dimensions, this flow would rotate both the position and the momentum about some axis.)

3. Given $A \in \mathfrak{so}(n)$, define an observable $F \in C^\infty(\mathbb{R}^{2n}$ by

$$F(q,p) = \sum_{i,j=1}^{n} A_{ij}(q_i p_j - q_j p_i).$$

Show that some multiple of F generates the flow ϕ defined above.

(I say 'some multiple' because you may need a factor of $\frac{1}{2}$ or a minus sign or something in front of F to make this calculation work. I leave that to you!)

The moral: The observable that generates the flow ϕ is called **angular momentum in the A direction**. But beware: A is not a vector in $\mathbb{R}^n$! It's a matrix in $\mathfrak{so}(n)$! For $n = 3$ we have an isomorphism

$$\mathfrak{so}(n) \cong \mathbb{R}^n$$

so we can talk about angular momentum in some direction $v \in \mathbb{R}^n$. But, this is not true in any other dimension (except $n = 0$)!

4. When $n = 3$, the observable

$$F(q,p) = q_1 p_2 - q_2 p_1$$

is usually called **angular momentum in the z direction** and denoted J_z. What flow does this observable generate?