Classical Mechanics Homework
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Angular Momentum and Rotations

1. Let A^* be skew-adjoint. Show that e^{tA} is orthogonal.

Let's take a broader view and work in the setting of Hilbert Spaces and bounded linear operators. Our first task is to show that the adjoint operator is continuous. To see this, let A be a bounded linear operator. Then

$$\|Ax\|^2 = (Ax, Ax) = (x, A^*Ax) \leq \|A^*A\| \leq \|A^*\|\|A\|,$$

where the second to last inequality follows by Cauchy-Schwartz (x is any unit vector). Then it follows (after taking a supremum) that $\|A\| \leq \|A^*\|$. Reversing the roles of A and A^* and using the fact that $(A^*)^* = A$ gives the reverse inequality. Hence, we see that for any bounded linear operator, $\|A\| = \|A^*\|$. Now let $A_n \to A$ be any convergent sequence of operators. We have that

$$\|A_n^* - A^*\| = \|(A - A_n)^*\| = \|A_n - A\|$$

and so $A_n^* \to A^*$; the adjoint is therefore a continuous map on the space of bounded linear operators.

Now, the exponential is defined as a convergent infinite series of operators, so we have by continuity:

$$\left(e^{tA}\right)^* = e^{(tA)^*} = e^{tA^*}.$$

If then $A^* = -A$ we have that

$$\left(e^{tA}\right)^* e^{tA} = e^{-tA}e^{tA} = e^{0A} = I,$$

and so, in this case, exponential of A is orthogonal.

2. Let $A \in \mathfrak{so}(n)$ and ϕ defined by:

$$\phi(t, q, p) = \left(e^{tA}q, e^{tA}p\right).$$

Show that ϕ is a flow.

We begin by showing that ϕ_0 is the identity:

$$\phi_0(q, p) = \left(e^{0A}q, e^{0A}p\right) = (Iq, Ip) = (q, p).$$

(*The fact that the exponential of the zero operator is the identity follows directly from the power series definition of the exponential.*) Now we show the additive property of the flow:

$$\phi_{t+s}(q, p) = \left(e^{(t+s)A}q, e^{(t+s)A}p\right) = \left(e^{tA}\left(e^{sA}q\right), e^{tA}\left(e^{sA}p\right)\right) = \phi_t(\phi_s(q, p)).$$

Finally, to see that ϕ is smooth, we note that the map $f : \mathbb{R} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ given by

$$f(t, A) = e^{tA}.$$

is smooth: the components of the matrix e^{tA} are simply uniformly convergent—because the series defining the exponential is *norm* convergent—power series in t and the components of A, and hence the individual component functions of f are smooth. Thus f is smooth, and therefore its restriction to the smooth submanifold $\mathbb{R}\times\mathfrak{so}(n)$ of $\mathbb{R}\times\mathbb{R}^{n^2}$ must be smooth. Whence, ϕ, whose two component functions are both given by this restriction of f, is smooth as a map

$$\phi : \mathbb{R} \times \mathfrak{so}(n) \to \mathfrak{so}(n).$$

(The fact that the range is given as $\mathfrak{so}(n)$, as opposed to $\mathbb{R}^{n^2}$, follows from exercise 1.)

3. For a skew adjoint matrix A, show that the observable

$$F(q,p) = \frac{1}{2}\sum a_{ij}(q_i p_j - q_j p_i)$$

generates the flow from the previous exercise.

First note that from the skew-adjointness of A we have

$$\partial_{q_k} F = \sum_i a_{ki} p_i = (Ap)_k \qquad \text{and} \qquad \partial_{p_k} F = -\sum_i a_{ki} q_i = -(Aq)_k,$$

from which it follows that the vector field generated by F is

$$\{F,\cdot\} = \sum_k (Ap)_k \partial_{p_k} + (Aq)_k \partial_{q_k},$$

and so the flow $\phi_t = (\psi(t), \varphi(t))$ is determined by the following two systems of ODE's:

$$\begin{cases} \psi'(t) = A\psi(t) \\ \psi(0) = p \end{cases} \qquad \text{and} \qquad \begin{cases} \varphi'(t) = A\varphi(t) \\ \varphi(0) = q. \end{cases}$$

But the solutions to these systems come easily as:

$$\psi(t) = e^{tA}p \qquad \text{and} \qquad \varphi(t) = e^{tA}q.$$

This yields the desired flow.

4. Consider the observable

$$F(q,p) = q_1p_2 - q_2p_1.$$

Determine the flow.

In light of the previous exercise, the skew-adjoint matrix associated with this observable is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Whence the flow is given by:

$$\phi_t(q,p) = \left(e^{tA}q, e^{tA}p\right).$$

Now, notice that

$$Ax = -ix$$

where on the right the vector x is treated as the complex number $x_1 + ix_2$, so that we have (continuing to play loose with the identification between $\mathbb{R}^2$ and $\mathbb{C}$):

$$e^{tA}x = e^{-it}x = R_t x$$

where R_t denotes *clockwise* rotation through an angle of t radians in $\mathbb{R}^2$. Thus we have that

$$\phi_t(q,p) = (R_t q, R_t p)$$

and the flow is simultaneous clockwise rotation in the q and p planes.