Classical Mechanics, Lecture 11
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1 How Hamiltonians Generate Time Evolution

We have seen how any observable H on a Poisson manifold X gives rise a vector field v_H which, if integrable, generates a flow ϕ. When H has the physical meaning of 'energy', we call it a **Hamiltonian** and call this flow **time evolution**. Let's see how this works in some examples we've already studied.

Example: The simple harmonic oscillator.
Here the configuration space is $\mathbb{R}$ and the phase space is $X = T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \ni (q,p)$. We have Newton's second law

$$m\frac{d^2}{dt^2}q(t) = -kq(t)$$

Let us set $m = k = 1$ to simplify the math a bit:

$$\frac{d^2q}{dt^2} = -q$$

Since $p = m\dot{q}$, with $m = 1$ this equation gives Hamilton's equations

$$\frac{dq}{dt} = p$$

$$\frac{dp}{dt} = -q$$

with solution:

$$q(t) = A\cos t + B\sin t$$

$$p(t) = -A\sin t + B\cos t$$

where $A = q(0)$ and $B = p(0)$. So we get a flow:

$$\phi \colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

defined by

$$(t, q(0), p(0)) \mapsto (q(t), p(t))$$

describing the time evolution of a point (q,p) in the phase space $\mathbb{R}^2$. This flow is rotation clockwise by an angle t over time t.

picture of $\mathbb{R}^2$ with flow lines and rotated point.

Now let's obtain this flow using the Poisson approach. So, we will remember the Hamiltonian for the harmonic oscillator and work out the vector field v_H generated by that, and see that it does indeed generate this flow.

The energy of the harmonic oscillator is:

$$E = \frac{1}{2}m\dot{q}^2 + \frac{k}{2}q^2$$

or with $m = k = 1$

$$E = \frac{1}{2}(q^2 + \dot{q}^2)$$

so the Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$ is

$$H(q,p) = \frac{1}{2}(q^2 + p^2)$$

(Note: flow moves along level sets of H)

Our prescription says: see what vector field H generates and then see what flow that vector field generates. H generates the vector field

$$\begin{aligned} v_H &= \{H, \cdot\} \\ &= \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \end{aligned}$$

(or $\left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$ in the old-fashioned notation for vector fields on $\mathbb{R}^2$.)

With our H this is:

$$v_h = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$$

picture of this vector field

If

$$\phi\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$

is the flow generated by v_H, then by definition

$$\frac{d}{dt}\phi_t(q,p) = v_H(\phi_t(q,p))$$

Notation: $\phi_t(q,p) := \phi(t,q,p)$. For short, let's write

$$\phi_t(q,p) = (q(t), p(t)) \in \mathbb{R}^2.$$

So, our equation becomes:

$$\begin{aligned} (\dot{q}(t), \dot{p}(t)) &= v_H(q(t), p(t)) \\ &= (p(t), -q(t)) \end{aligned}$$

and this is Hamilton's equations:

$$\dot{q}(t) = p(t)$$
$$\dot{p}(t) = -q(t)$$

so we must have:

$$(q(t), p(t)) = (\cos(t)q + \sin(t)p, -\sin(t)q + \cos(t)p)$$

which is the flow we had before.

Example: A particle in a potential in $\mathbb{R}^n$.
Now our configuration space is $\mathbb{R}^n$ and so the phase space is

$$X = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \ni (q,p)$$

The potential energy is $V(q)$ where $V \in C^\infty(\mathbb{R}^n)$. The kinetic energy is $\frac{1}{2}mv^2 = \frac{p^2}{2m}$ since $p = mv$. So the Hamiltonian is

$$H(q,p) = \frac{p^2}{2m} + V(q).$$

This generates the vector field

$$\begin{aligned} v_H &= \{H, \cdot\} \\ &= \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \\ &= \sum_{i=1}^n \frac{p_i}{m}\frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i}\frac{\partial}{\partial p_i} \end{aligned}$$

or in old-fashioned notation for vector fields on $\mathbb{R}^{2n}$:

$$\left(\frac{p_1}{m}, \frac{p_2}{m}, \ldots, \frac{p_n}{m}, -\frac{\partial V}{\partial q_1}, \ldots, -\frac{\partial V}{\partial q_n}\right)$$

The flow generated by this:

$$\phi_t(q,p) = (q(t), p(t))$$

satisfies:

$$\frac{d}{dt}(q(t),p(t)) = \left(\frac{p(t)}{m}, -\nabla V(q(t))\right)$$

which we have seen before:

$$m\frac{d}{dt}q(t) = p(t)$$

$$\frac{d}{dt}p(t) = -\nabla V(q(t))$$

2 The Galilei Group

In our homework we have seen how the Galilei group acts on the phase space of a point particle in $\mathbb{R}^n$. Let us see how some of the symmetries come from observables.

Example: How momentum generates spatial translations.
Consider a particle in $\mathbb{R}^n$, so $X = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. Consider the flow "spatial translation by an amount $s \in \mathbb{R}$ in the $u \in \mathbb{R}^n$ direction". Call this flow $\phi\colon \mathbb{R} \times X \to X$; it is given by

$$\phi_s(q,p) = (q + su, p)$$

The vector field generating this flow, say v, has:

$$\begin{aligned} v(q,p) &= \left.\frac{d}{ds}\phi_s(q,p)\right|_{s=0} \\ &= \left.\frac{d}{ds}(q+su,p)\right|_{s=0} \\ &= (u,0) \end{aligned}$$

Next, let's find an observable $F \in C^\infty(X)$ that generates this v:

$$v_F = v$$

Note:

$$\begin{aligned} v &= \sum_{i=1}^n u_i \frac{\partial}{\partial q_i} + 0\frac{\partial}{\partial p_i} \\ &= \sum_{i=1}^n u_i \frac{\partial}{\partial q_i} \end{aligned}$$

whereas

$$\begin{aligned} v_F &= \{F, \cdot\} \\ &= \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i} \end{aligned}$$

so

$$\frac{\partial F}{\partial q_i} = 0 \text{ and } \frac{\partial F}{\partial p_i} = u_i$$

So

$$F(q,p) = p \cdot u + c$$

for any $c \in \mathbb{R}$, which we'll take to be 0, so we say "spatial translations in the u dirction are generated by F", where F is "momentum in the u direction".

Example: What observable generates Galilei boosts by an amount s in the u direction (where $u \in \mathbb{R}^n$)?
Here the flow is:

$$\phi_s(q,p) = (q, p + msu)$$

i.e., ϕ_s does not change the position but it adds su to the velocity! This is generated by:

$$\begin{aligned} v &= \left. \frac{d}{ds}\phi_s(q,p) \right|_{s=0} \\ &= (0, mu) \\ &= \sum mu_i \frac{\partial}{\partial p_i} \end{aligned}$$

and this equals

$$v_F = \sum \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}$$

for

$$F = -mu \cdot q.$$

So, there's a nice pattern:
for a single particle in $\mathbb{R}^n$:
Galilei boosts in the direction $u \in \mathbb{R}^n$ are generated by $-mq \cdot u$.
Translations in the direction $u \in \mathbb{R}^n$ are generated by $p \cdot u$.

For a bunch of particles in $\mathbb{R}^n$, this generalizes:
Galilei boosts in the direction $u \in \mathbb{R}^n$ are generated by $-\sum m_i q_i \cdot u$, i.e. $-u \cdot$ (total mass $\times$ center of mass).
Translations in the direction $u \in \mathbb{R}^n$ are generated by $p \cdot u$, where $p = \sum p_i$ is the total momentum (sum over particles: $p_i \in \mathbb{R}^n$ is the momentum of the i^{th} particle).