Classical Mechanics, Lecture 13

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1 Lie Algebras

Lie groups are groups that are also manifolds. These let us describe continuous symmetries in physics. A classic example is SO(n) - the group of rotations of $\mathbb{R}^n$. Every Lie group has a 'Lie algebra', which lets us do calculations more easily. The Lie algebra is a vector space, so it lets us use linear algebra to study Lie groups.

For example: SO(n) has a Lie algebra

$$\mathfrak{so}(n) = \{A : A \text{ is an } n \times n \text{ real matrix with } A^* = -A\}$$

The point is that given $A \in \mathfrak{so}(n)$,

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \in \mathrm{SO}(n), \quad (t \in \mathbb{R})$$

In fact, any Lie group G has a Lie algebra $\mathfrak{g}$ and there is a map called exponentiation

$$\exp\colon \mathfrak{g} \to G$$

such that

$$\exp((s+t)A) = \exp(sA)\exp(tA).$$

For example, if $\mathfrak{g} = \mathbb{R}$ (a vector spce) and $G = \mathbb{R}^+$ (the Lie group of positive real numbers, with multiplication as the group operation), then

$$\exp : \mathbb{R} \to \mathbb{R}^+$$

is the usual exponential function and I am just saying $\exp(s+t) = \exp(s)\exp(t)$. Here a slide rule (or table of logarithms) converts problems in the Lie group into problems in the Lie algebra.

Definition 1 *A* **Lie algebra** *is a (real) vector space* $\mathfrak{g}$ *equipped with an operation*

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the **Lie bracket** *which has the following properties:*

1. *bilinearity*
2. *antisymmetry*
3. *the Jacobi identity*

We've seen one type of Lie algebra so far: if X is a Poisson manifold, $C^\infty(X)$ is a Lie algebra with the Poisson bracket as its bracket. But in fact, *every* manifold gives a Lie algebra, in a different way:

Theorem 2 *Given any manifold X, let* $\mathrm{Vect}(X)$ *be the set of vector fields on X. This becomes a Lie algebra by:*

$$(\alpha v)f = \alpha(v(f))$$
$$(v+w)f = vf + wf$$
$$[v,w]f = v(wf) - w(vf)$$

where $\alpha \in \mathbb{R}$ and $f \in C^\infty(X)$.

Sketch of proof:
Just calculate to check the Lie algebra axioms. For example: a vector field $v \in \mathrm{Vect}(X)$ is a linear map

$$v\colon C^\infty(X) \to C^\infty(X)$$

satisfying $v(fg) = v(f)g + fv(g)$.

Why does $v, w \in \mathrm{Vect}(X) \Rightarrow [v,w] \in \mathrm{Vect}(X)$?

$$\begin{aligned}
[v,w](fg) &= vw(fg) - wv(fg) \\
&= v(w(f)g + fw(g)) - w(v(f)g + fv(g)) \\
&= (vw(f))g + w(f)v(g) + v(f)w(g) + f(vw(g)) - wv(f)g - v(f)w(g) - w(f)v(g) - f(wv(g)) \\
&= ([v,w]f)g + f([v,w]g).
\end{aligned}$$

Why does $[\cdot,\cdot]$ satisfy the Jacobi identity?

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]]$$

$$\begin{aligned}
([[u,v],w] + [v,[u,w]])f &= (uvw - vuw - wuv + wvu + vuw - vwu - uwv + wuv)f \\
&= [u,[v,w]]f
\end{aligned}$$

Etc.. . .

This Lie algebra $\mathrm{Vect}(X)$ is very related to the Poisson algebra $C^\infty(X)$ when X is a Poisson manifold:

Theorem 3 *If X is a Poisson manifold and $f, g \in C^\infty(X)$ then*

$$[v_f, v_g] = v_{\{f,g\}}.$$

Proof - For any $h \in C^\infty(X)$,

$$\begin{aligned}
v_{\{f,g\}}h &= \{\{f,g\},h\} \\
&= \{f\{g,h\}\} + \{\{f,h\},g\} \\
&= \{f,\{g,h\}\} - \{g,\{f,h\}\} \\
&= v_f v_g h - v_g v_f h \\
&= [v_f, v_g]h
\end{aligned}$$

Later we'll define a 'homomorphism' between Lie algebras, and the theorem we just proved will say that

$$v \mapsto v_f$$

is a homomorphism from the Lie algebra $C^\infty(X)$ to the Lie algebra $\mathrm{Vect}(X)$.