Classical Mechanics, Lecture 14

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lecture by John Baez

notes by Alex Hoffnung

1 The Lie Algebra of a Lie Group

We've seen some Lie groups already, like $\mathrm{SO}(n)$. Now it's time for a formal definition:

Definition 1 *A* **Lie group** *is a group G that is also a manifold such that multiplication*

$$m\colon G \times G \to G$$

$$(g,h) \mapsto gh$$

and inverse map

$$\mathrm{inv}\colon G \to G$$

$$g \mapsto g^{-1}$$

are smooth maps.

Every Lie group has a Lie algebra. To get our hands on this, given a Lie group G we define a vector space

$$\mathfrak{g} = T_1 G$$

where $1 \in G$ is the identity. We want to make this into a Lie algebra and construct an 'exponentiation' map

$$\exp\colon \mathfrak{g} \to G.$$

But we need to start slowly, and build up some basic tools first.

Recall that given a smooth map between manifolds $\psi\colon X \to N$ we get a linear map called **pushing forward along** ψ:

$$\psi_* = d\psi_x\colon T_x X \to T_{psi(x)} N$$

picture of pushforward of tangent vector

We can define ψ_* by saying it sends $[\gamma] \in T_x X$ where $\gamma\colon \mathbb{R} \to X$ with $\gamma(0) = x$, to $[\psi\gamma] \in T_{\psi(x)} N$.

Check:

$$(1_X)_* = 1_{T_x X}$$

and if $\psi\colon M \to N$ and $\psi\colon N \to P$ then

$$\phi_* \psi_* = (\phi\psi)_*$$

This idea lets us think of the $\mathfrak{g}$ as a space of vector fields called 'left-invariant' vector fields:

Theorem 2 *$\mathfrak{g}$ is isomorphic to the vector space of* **left-invariant vector fields** *on G, i.e. vector fields $v \in Vect(G)$ such that*

$$(L_g)_* v(h) = v(gh), \quad \forall g, h \in G$$

where **left multiplication by** *g is:*

$$L_g\colon G \to G$$

$$h \mapsto gh.$$

The isomorphism goes as follows:

$$\{\textit{left-invariant vector fields}\} \to \mathfrak{g}$$

$$v \mapsto v(1)$$

Before we prove this, let's draw a picture to explain it!

picture of $U(1)$ in the complex plane with tangent space at identity before and after left multiplication

A left-invariant vector field on $U(1)$ is one of constant length, always pointing the same way. Given $x \in T_1 G$ we get a left-invariant vector field V on G by

$$v(g) = (L_g)_* x$$

Now let's prove the theorem:

Proof - We construct an inverse map:

$$\mathfrak{g} \to \{\text{left-invariant vector fields}\}$$

$$x \mapsto v^x$$

where

$$v^x(h) = (L_h)_* x \in T_h G.$$

Let's check that v^x is left-invariant:

$$\begin{aligned} (L_g)_* v^x(h) &= (L_g)_*(L_h)_* x \\ &= (L_g L_h)_* x \\ &= (L_{gh})_* x \\ &= v^x(gh). \end{aligned}$$

Next check it is an inverse map. First: start with a left-invariant w, turn it into $w(1) \in \mathfrak{g}$, then turn that back into a left-invariant vector field $v^{w(1)}$. Check: $w = v^{w(1)}$.

$$\begin{aligned} v^{w(1)}(h) &= (L_h)_* w(1) \\ &= w(h). \end{aligned}$$

Second: start with $x \in \mathfrak{g}$, turn it into a left-invariant vector field v^x, then turn that back into $v^x(1) \in \mathfrak{g}$. Check: $x = v^x(1)$.

$$\begin{aligned} v^x(1) &= (L_1)_* x \\ &= (1_G)_* x \\ &= x. \end{aligned}$$

picture of Lie-algebra element going to left-invariant vector field on the circle and vice versa

We henceforth use this isomorphism to freely think of $\mathfrak{g}$ *either* as $T_1 G$ or as the space of all left-invariant vector fields on G. We use this to define a bracket operation on $\mathfrak{g}$, using the fact that $Vect(G)$ is a Lie algebra. Using $\mathfrak{g} \subseteq Vect(G)$ to make $\mathfrak{g}$ into a Lie algebra we just need:

Lemma 3 *If $v, w \in Vect(G)$ are left-invariant, so is $[v, w]$.*

Proof - For this we will use a general fact: if $\phi\colon M \to N$ is a diffeomorphism, then given $v \in Vect(M)$

picture of push-forward of vector

If ϕ is a diffeomorphism there is a unique $x \in M$ mapping to any $y \in N$ (namely $\phi^{-1}(y)$), so we can define a vector field $\phi_* v \in Vect(N)$ by:

$$(\phi_* v)(y) = \phi_* v(\phi^{-1}(y))$$

In fact, if $\phi: M \to N$ is a diffeomorphism and $v, w \in Vect(M)$ then:

$$\phi_*[v, w] = [\phi_* v, \phi_* w].$$

In particular, if v and w are left-invariant vector fields on G,

$$(L_g)_* v = v$$

$$(L_g)_* w = w$$

so

$$\begin{aligned} (L_g)_*[v, w] &= [(L_g)_* v, (L_g)_* w] \\ &= [v, w] \end{aligned}$$

so $[v, w]$ is left-invariant.

Now that $\mathfrak{g}$ is a Lie algebra let's define the exponential map

$$exp: \mathfrak{g} \to G.$$

We will use a sort of hard fact

Theorem 4 *If G is a Lie group, every left-invariant $v \in Vect(G)$ is integrable.*

(By the way: If M is a compact manifold, then every $v \in Vect(M)$ is integrable.)

Given this, any $v \in \mathfrak{g}$ thought of as a left-invariant vector field, generates a flow:

$$\phi: \mathbb{R} \times G \to G$$

$$(t, g) \mapsto \phi_t(g)$$

and we define:

$$\exp(tv) = \phi_t(1) \in G$$

Example: $G = \mathrm{U}(1)$

picture of circle with left-invariant vector field

This generates a flow where $\phi_t: U(1) \to U(1)$ is rotation by the angle $\frac{tv}{i} \in \mathbb{R}$. Note: rotation by $\frac{tv}{i}$ is the same as multiplication by $e^{tv} \in U(1)$. So

$$\exp(tv) = \phi_t(1) = e^{tv} 1 = e^{tv} \in U(1)$$

Whew!

Example: $G = \mathrm{SO}(n)$, rotation group of $\mathbb{R}^n$.
This is a group of matrices, with matrix multiplication as the group operation. So $\mathrm{SO}(n)$ sits in the vector space of $n \times n$ matrices:

artist's depiction of $\mathrm{SO}(n)$ with its Lie algebra $\mathfrak{so}(n)$

In this case the Lie algebra

$$\mathfrak{so}(n) = \{A: \mathbb{R}^n \to \mathbb{R}^n : A \text{ linear and } A^* = -A\}$$

and

$$\exp: \mathfrak{so}(n) \to \mathfrak{so}(n)$$

is given by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

With this definition of $exp\colon \mathfrak{g} \to G$ we can check:

$$\exp(0) = 1 \in G$$

(easy, since a flow has $\phi_0 = 1$) and:

$$\exp((s+t)v) = exp(sv)exp(tv)$$

(trickier, but we use $\phi_{s+t} = \phi_s \phi_t$ and some more).

2 Actions of Lie Groups

What are Lie groups good for? They 'act' on manifolds!

Definition 5 *If G is a Lie group and X is some manifold, an* **action** *of G on X is a smooth map*

$$\phi\colon G \times X \to X$$

$$(g, x) \mapsto \phi(g)x$$

such that:

$$\phi(gh)x = \phi(g)\phi(h)x, \quad \forall g, h \in G, \forall x \in X$$

and:

$$\phi(1)x = x.$$

(These imply: $\phi(g^{-1}) = \phi(g)^{-1}$).

Example: A flow is the same as an action of $G = \mathbb{R}$.

Example: The Euclidean group $E(n)$ is a Lie group and it acts on $\mathbb{R}^n$ ("space").

Example: The Galilei group $G(n+1)$ is a Lie group and it acts on $\mathbb{R}^{n+1}$ ("spacetime"). We also saw how $G(n+1)$ acts on the phase space of a free particle in $\mathbb{R}^n$, $X = \mathbb{R}^n \times \mathbb{R}^n \ni (q, p)$.

We have seen how symmetries in $G(n+1)$ are related to conserved quantities, certain functions on the Poisson manifold X. We have seen that (almost) any single function on a Poisson manifold generates a flow, i.e. an action of $\mathbb{R}$. (The vector field might fail to be integrable.)

When does a collection of functions on a Poisson manifold X give rise to an action of a Lie group G on X? Or conversely, which group actions on X give rise to a bunch of functions on X? These are important questions that we'll begin to tackle next time.