Classical Mechanics, Lecture 15
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1 Group Actions Preserving Structures

We've said a bit about group actions, but I haven't explained the most important part: how any structure on a manifold gives a group action!

Given a manifold X with some extra structure (for example: a Riemannian metric - a very nice distance function), you can ask: which diffeomorphisms $f: X \to X$ preserves this extra structure? Such diffeomorphisms form a group! We get lots of interesting groups, including a lot of Lie groups.
Example: Take $X = \mathbb{R}^n$ and let the extra structure be the structure of the vector space. Then $f: \mathbb{R}^n \to \mathbb{R}^n$ preserves this extra structure iff:

- $f(0) = 0$
- $f(x + y) = f(x) + f(y)$
- $f(\alpha x) = \alpha f(x)$

We say f is **linear**. We get a group of diffeomorphisms $f: X \to X$ preserving the vector space structure:

$$\begin{aligned} G &= \mathrm{GL}(n) \\ &= \{f: \mathbb{R}^n \to \mathbb{R}^n : f \text{is linear and invertible}\} \end{aligned}$$

Example: Let $X = \mathbb{R}^n$ but with its vector space structure and its usual metric. The group preserving this structure is

$$\begin{aligned} G &= \mathrm{O}(n) \\ &= \{f \in \mathrm{GL}(n) : ||f(x)|| = ||x||\} \end{aligned}$$

If we also demand that our transformations preserve an 'orientation' on $\mathbb{R}^n$, we get a smaller group that excludes reflections: $\mathrm{SO}(n)$.
Example: Let $X = \mathbb{R}^n$ with its usual metric. The group preserving this structure is the Euclidean group

$$\begin{aligned} G &= E(n) \\ &= \mathrm{O}(n) \ltimes \mathbb{R}^n \end{aligned}$$

In other words, any $f \in E(n)$ can be written as:

$$f(x) = Rx + v$$

for $R \in \mathrm{O}(n)$, $v \in \mathbb{R}^n$, but

$$ff'(x) = R(R'x + v') + v = RR'x + Rv' + v$$

with $f, f' \in E(n)$, instead of merely $RR'x + v + v'$ which we would have in $\mathrm{O}(n) \times \mathbb{R}^n$.

We can turn this game around: given a group (e.g. a Lie group) G acting on a manifold X we can ask: what structure on X does this group preserve? We have seen that the Galilei group $G(n+1)$ acts on $\mathbb{R}^n \times \mathbb{R}$: it is the group generated by:

- spatial translations ($\mathbb{R}^n$)
- rotations and reflections ($\mathrm{O}(n)$)
- time translations ($\mathbb{R}$)
- Galilei boosts ($\mathbb{R}^n$)

What structure on spacetime ($\mathbb{R}^n \times \mathbb{R}$) is preserved precisely by this action of $G(n+1)$?

1. "Simultaneous distances" - i.e. distance between points (x, t) and (x', t), namely $||x - x'||$.
2. "Time intervals" - the time intervals between (x, t) and (x', t') is $t - t'$.
3. "Lines" - all lines in $\mathbb{R}^{n+1}$ are mapped to lines. Or: it suffices to demand that the paths of free particles are mapped to paths of free particles.

When Einstein came up with special relativity, he saw that the Galiliei group is not really the correct group of symmetries of spacetime — the correct group is much simpler! The correct group preserves 'spacetime distances', or more precisely

$$d((x, y, z, t) - (x', y', z', t')) = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2 - c^2(t - t')^2}$$

where c is the speed of light. This group 'approaches' the Galilei group as $c \to \infty$.