Classical Mechanics, Lecture 16

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1 Symmetries and Observables

Now let's dig deeper into the relation between symmetries and observables. We want to say exactly when a group acting as symmetries of a classical system gives a bunch of observables. We will call such such a group action a 'Hamiltonian action', because the simplest example is how the Hamiltonian of any classical system is related to time translation symmetry.

The concept of Hamiltonian action involves a trio of Lie algebras and Lie algebra homomorphisms. Let us introduce them one at a time! First, for any manifold X, the space of vector fields $\mathrm{Vect}(X)$ is a Lie algebra. Second, if X is a Poisson manifold, the algebra of observables $C^\infty(X)$ is also a Lie algebra. Third, we we have a map

$$\beta \colon C^\infty(X) \to \mathrm{Vect}(X)$$

$$f \mapsto v_f = \{f, \cdot\}$$

As we already hinted, this map is a 'Lie algebra homomorphism':

Definition 1 *If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a map $f \colon \mathfrak{g} \to \mathfrak{h}$ is a* **Lie algebra homomorphism** *if f is linear and it preserves the Lie bracket, as follows:*

$$\alpha([x,y]) = [\alpha(x), \alpha(y)]$$

for all $x, y \in \mathfrak{h}$.

Indeed, $\beta \colon C^\infty(X) \to \mathrm{Vect}(X)$ is linear since the Poisson bracket is bilinear, and we have seen that it preserves the bracket:

$$v_{\{f,g\}} = [v_f, v_g]$$

Remember why: this is just the Jacobi identity for $\{\cdot,\cdot\}$:

$$\begin{aligned} v_{\{f,g\}} &= \{\{f,g\},\cdot\} \\ &= \{f,\{g,\cdot\}\} - \{g,\{f,\cdot\}\} \\ &= v_f v_g \cdot - v_g v_f \cdot \\ &= [v_f, v_g] \cdot \end{aligned}$$

Next, it turns out that whenever we have a Lie group G acting on a manifold X, we get another Lie algebra homomorphism, from the Lie algebra of G to $\mathrm{Vect}(X)$:

Theorem 2 *Suppose G is a Lie group, X is a manifold, and*

$$A \colon G \times X \to X$$

$$(g, x) \mapsto A(g)x$$

is an action. Then we get a Lie algebra homomorphism

$$\alpha \colon \mathfrak{g} \to \mathrm{Vect}(X).$$

Sketch of Proof: We will define α but not show $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ or linearity of α. Given $x \in \mathfrak{g}$ we form $\exp(tx) \in G$, which gives a flow on X:

$$\phi\colon \mathbb{R} \times X \to X$$

$$(t,x) \mapsto A(\exp(tv))x$$

Why is this a flow? Check:

1. $A(\exp(0v))x = A(1)x = x$;
2. $A(\exp(t+s)v)x = A(\exp(tv)\exp(sv))x = A(\exp(tv))A(\exp(sv))x$.

Then to get $\alpha(v) \in \mathrm{Vect}(X)$ we just differentiate this flow:

$$\alpha(v)(x) = \frac{d}{dt} A(\exp(tv))x \mid_{t=0} \in T_x X, \quad \forall x \in X$$

Let's look at an example:

Example: SO(3) acts on $\mathbb{R}^3$ in an obvious way, so we get

$$\alpha\colon \mathfrak{so}(3) \to \mathrm{Vect}(\mathbb{R}^3).$$

For example, take

$$e_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is called the "generator of rotations around the z-axis" since

$$\exp(te_z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which describes rotation around z axis, which as t varies gives a flow $\phi\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\phi_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Differentiating this we get a vector field:

picture of flow around z-axis

In equations this vector field is

$$\frac{d}{dt}\phi_t \begin{pmatrix} x \\ y \\ z \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

Now, suppose X is a Poisson manifold and we have an action $A\colon G \times X \to X$. Then we get *two* Lie algebra homomorphisms:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\alpha"] \arrow[dr, dashed, "\gamma"'] & & \mathrm{Vect}(X) \\ & C^\infty(X) \arrow[ur, "\beta"'] & \end{tikzcd}$$

We have Lie algebra homomorphisms α and β, where

$$\beta(f) := v_f := \{f, \cdot\}$$

and we say the action A is **Hamiltonian** if we can find a Lie algebra homomorphism γ such that $\alpha = \beta\gamma$. Such a γ gives an observable $\gamma(v) \in C^\infty(X)$ for any **infinitesimal symmetry** $v \in \mathfrak{g}$, such that

$$\alpha(v) = \beta(\gamma(v))$$

i.e.

$$\frac{d}{dt} A(\exp(tv))x|_{t=0} = \{\gamma(v), \cdot\}$$

i.e. the observable $\gamma(v)$ generates the flow $(t, x) \mapsto A(\exp(tv))x$. In this case we have a nice map from (infinitesimal) symmetries to observables!

Example: $G = \mathrm{SO}(3)$ acts on $\mathbb{R}^3$, the configuration space of a particle in $\mathbb{R}^3$, and thus it acts on the phase space

$$X = T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3 \ni (q, p)$$

In detail: we have

$$A \colon G \times X \to X$$

$$(g, q, p) \mapsto (gq, gp)$$

Is this action Hamiltonian? Yes. What is

$$\gamma \colon \mathfrak{so}(3) \to C^\infty(X)?$$

In 3-dimensions, we have $\mathfrak{so}(3) \cong \mathbb{R}^3$ where the standard basis of $\mathbb{R}^3$ corresponds to

$$e_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$e_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Check:

$$[e_x, e_y] = e_z$$

and cyclic permutations - so $[\cdot, \cdot]$ in $\mathfrak{so}(3)$ corresponds to $\times$ in $\mathbb{R}^3$.
Identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ using this isomorphism. Then

$$\gamma(v) = v \cdot J$$

where

$$J = q \times p \in \mathbb{R}^3$$

is the angular momentum. Let's check that this works:

$$\alpha = \beta\gamma$$

Let's just check

$$\alpha(e_z) = \beta(\gamma(e_z))$$

Left side:

$$\begin{aligned} \alpha(e_z)(q,p) &= \frac{d}{dt}(\exp(te_z)q, \exp(te_z)p)\Big|_{t=0} \\ &= (-q_2, q_1, 0, -p_2, p_1, 0) \in \mathbb{R}^3 \times \mathbb{R}^3 \end{aligned}$$

Right side:

$$\begin{aligned} \gamma(e_z) &= e_z \cdot J \\ &= e_z \cdot (q \times p) \\ &= q_1 p_2 - q_2 p_1 \end{aligned}$$

so

$$\begin{aligned} \beta(e_z) &= \{q_1p_2 - q_2p_1, \cdot\} \\ &= \sum_{i=1}^{3} \frac{\partial}{\partial p_i}(q_1p_2 - q_2p_1)\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i}(q_1p_2 - q_2p_1)\frac{\partial}{\partial p_i} \\ &= -q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} + 0\frac{\partial}{\partial q_3} - p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2} + 0\frac{\partial}{\partial p_3} \end{aligned}$$

which is the same vector field in modern notation.

Example: The Euclidean group $E(n)$ acts on $\mathbb{R}^n$ and thus on $X = T^*\mathbb{R}^n$, and this action is Hamiltonian.

Example: The Galilei group $G(n+1)$ acts on $X = T^*\mathbb{R}^n$, and this action is not Hamiltonian!

There is an obvious candidate for

$$\gamma \colon \mathfrak{g}(n+1) \to C^\infty(T^*\mathbb{R}^n)$$

which sends :

1. standard basis vectors of $\mathfrak{so}(n)$ to components of angular momentum $J_{ij} = q_i p_j - q_j p_i$
2. standard basis vectors of the spatial translation Lie algebra $\mathbb{R}^n$ to components of momentum p_i
3. standard basis vector of the time translation Lie algebra $\mathbb{R}$ to the Hamiltonian H
4. standard basis vectors of the Galilei boost Lie algebra $\mathbb{R}^n$ to components of mass times position, mq_i.

We indeed have

$$\alpha = \beta\gamma$$

in this case, but γ is not a Lie algebra homomorphism! Let $r, s \in \mathfrak{g}(n+1)$ be as follows:

r generates spatial translations in the first coordinate direction

i.e.

$$r = (0, (1, 0, \ldots, 0), 0, 0) = \mathfrak{so}(n) \oplus \mathbb{R}^n \oplus \mathbb{R} \oplus \mathbb{R}^n = (g)(n+1)$$

and

$$s \text{ generates boosts in the first coordinate direction}$$

i.e.

$$s = (0, 0, 0, (1, 0, 0, \ldots))$$

I claim:

$$\gamma([r, s]) \neq \{\gamma(r), \gamma(s)\}$$

First, let's see that $[r, s] = 0$. To do this, we ask: in $G(n+1)$ do spatial translations commute with boosts? Say $n = 1$:

$$\begin{tikzcd}
(t, x) \arrow[r, mapsto, "\text{translate by } a"] \arrow[d, mapsto, "\text{boost by} v"'] & (t, x + a) \arrow[d, mapsto, "\text{boost by } v"] \\
(t, x + tv) \arrow[r, mapsto, "\text{translate by} a"'] & (t, x + tv + a) = (t, x + a + tv)
\end{tikzcd}$$

These commute, so $[r, s] = 0$, since whenever Lie algebra elements generate commuting group elements $\exp(ar)$, $exp(vs)$ for all $a, v \in \mathbb{R}$ we have $[r, s] = 0$. But: $\{\gamma(r), \gamma(s)\} \neq 0$, as we will see.