Classical Mechanics, Lecture 17
March 6, 2008
lecture by John Baez
notes by Alex Hoffnung

## 1 Weakly Hamiltonian Group Actions

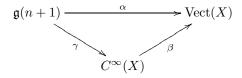
The phase space for the free particle in  $\mathbb{R}^n$  is  $X = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \ni (q, p)$ . We worked out the action of the Galilei group on X:

$$A: G(n+1) \times X \to X$$

so we have:

$$\alpha: \mathfrak{g}(n+1) \to \mathrm{Vect}(X)$$

and indeed we have this commuting diagram:



where  $\beta_f = \{f, \cdot\}$  and  $\gamma$  was described last time. But I claim: A is not Hamiltonian because there is no  $\gamma$  making this commute that is a Lie algebra homomorphism. We will see that the  $\gamma$  described last time does not work:

$$\gamma[r,s] \neq \{\gamma(r),\gamma(s)\}$$

for some  $r, s \in \mathfrak{g}(n+1)$ . We will take r to be the generator of translations in the  $1^{st}$  coordinate direction, and s to be the generator of Galilei boosts in that direction. E.g. if n = 1:

$$\exp(ar)(x,t) = (x+a,t), \quad \forall a \in \mathbb{R}$$

$$\exp(vs)(x,t) = (x+tv,t), \quad \forall v \in \mathbb{R}$$

We have [r, s] = 0 since:

**Lemma 1** If G is any Lie group and  $r, s \in \mathfrak{g}$ , then

$$[r,s] = 0$$

if and only if

$$\exp(ar)\exp(vs) = \exp(vs)\exp(ar), \quad \forall a, v \in \mathbb{R}$$

**Sketch of Proof**: We need just one direction of this if and only if, which can be shown roughly as follows:

$$\exp(ar)\exp(vs) = \exp(vs)\exp(ar)$$

so 
$$\left. \frac{\partial^2}{\partial a \partial v} \exp(ar) \exp(vs) \right|_{a,v=0} = \left. \frac{\partial^2}{\partial a \partial v} \exp(vs) \exp(ar) \right|_{a,v=0}$$

so rs = sr

so 
$$[r,s] \stackrel{?}{=} rs - sr = 0$$

This is legitimate if G is a group of matrices.

So  $\gamma[r,s] = 0$ . But  $\{\gamma(r), \gamma(s)\} \neq 0$ , since

$$\gamma(r) = p_1$$

(momentum generates translations), and

$$\gamma(s) = mq_1$$

(mass time position generates boosts), so

$$\begin{cases} \gamma(r), \gamma(s) \} &= \sum_{i=1}^{n} \frac{\partial}{\partial p_{i}} p_{1} \frac{\partial}{\partial q_{i}} m q_{1} - \frac{\partial}{\partial q_{i}} p_{1} \frac{\partial}{\partial p_{i}} m q_{1} \\ &= m \end{cases}$$

Here  $m \in C^{\infty}(X)$  is really the constant function equal to m at all points of X. But what vector field on X does this observable generate? What is  $v_m = \{m, \cdot\}$ ? It is zero! It generates this flow:

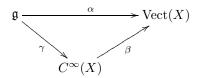
$$\phi: \mathbb{R} \times X \to X$$

$$(t,x)\mapsto x$$

All constant functions, or indeed, all locally constant functions f on any Poisson manifold give  $v_f = 0$ .

picture of phase space with two connected components

The problem is that this diagram:



 $\beta$  is not 1-1; it sends all locally constant functions to 0, so we can have

$$\beta \gamma[x, y] = [\beta \gamma(x), \beta \gamma(y)]$$

even though

$$\gamma[x,y] \neq [\gamma(x),\gamma(y)].$$

This also means that different choices of  $\gamma$  can make this diagram commute.

**Definition 2** If G is a Lie group acting on a Poisson manifold X:

$$A: G \times X \to X$$

we say A is weakly Hamiltonian if there exists a linear map

$$\gamma:\mathfrak{g}\to C^\infty(X)$$

such that

$$\alpha = \beta \gamma$$

with  $\gamma$  not necessarily a Lie algebra homomorphism.

In our example we have:

$$\{\gamma(x), \gamma(y)\} = \gamma[x, y] + c(x, y)$$

where c(x,y) is a locally constant function, so  $\{c(x,y),\cdot\}=0$ . In this situation we call c a "2-cocycle".

It turns out that weakly Hamiltonian actions of the Galilei group G(n+1) on the Poisson manifold  $T^*\mathbb{R}^n$  can completely classified: there is one for each  $m \in \mathbb{R}$ . This m specifies the cocycle ... but physically it is the **mass** of a particle in  $\mathbb{R}^n$ . So the concept of mass is inevitable ... even without gravity around!