Classical Mechanics, Lecture 17
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1 Weakly Hamiltonian Group Actions

The phase space for the free particle in $\mathbb{R}^n$ is $X = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \ni (q,p)$. We worked out the action of the Galilei group on X:

$$A: G(n+1) \times X \to X$$

so we have:

$$\alpha: \mathfrak{g}(n+1) \to \mathrm{Vect}(X)$$

and indeed we have this commuting diagram:

$$\begin{tikzcd} \mathfrak{g}(n+1) \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & \mathrm{Vect}(X) \\ & C^\infty(X) \arrow[ur, "\beta"'] & \end{tikzcd}$$

where $\beta_f = \{f, \cdot\}$ and γ was described last time. But I claim: A is not Hamiltonian because there is no γ making this commute that is a Lie algebra homomorphism. We will see that the γ descibed last time does not work:

$$\gamma[r,s] \neq \{\gamma(r), \gamma(s)\}$$

for some $r, s \in \mathfrak{g}(n+1)$. We will take r to be the generator of translations in the 1^{st} coordinate direction, and s to be the generator of Galilei boosts in that direction. E.g. if $n = 1$:

$$\exp(ar)(x,t) = (x+a,t), \quad \forall a \in \mathbb{R}$$

$$\exp(vs)(x,t) = (x+tv,t), \quad \forall v \in \mathbb{R}$$

We have $[r,s] = 0$ since:

Lemma 1 *If G is any Lie group and $r, s \in \mathfrak{g}$, then*

$$[r,s] = 0$$

if and only if

$$\exp(ar)\exp(vs) = \exp(vs)\exp(ar), \quad \forall a, v \in \mathbb{R}$$

Sketch of Proof: We need just one direction of this if and only if, which can be shown roughly as follows:

$$\exp(ar)\exp(vs) = \exp(vs)\exp(ar)$$

so

$$\frac{\partial^2}{\partial a \partial v} \exp(ar)\exp(vs)\bigg|_{a,v=0} = \frac{\partial^2}{\partial a \partial v} \exp(vs)\exp(ar)\bigg|_{a,v=0}$$

so

$$rs = sr$$

so

$$[r,s] \stackrel{?}{=} rs - sr = 0$$

This is legitimate if G is a group of matrices. ■

So $\gamma[r,s] = 0$. But $\{\gamma(r), \gamma(s)\} \neq 0$, since

$$\gamma(r) = p_1$$

(momentum generates translations), and

$$\gamma(s) = mq_1$$

(mass time position generates boosts), so

$$\begin{aligned} \{\gamma(r), \gamma(s)\} &= \sum_{i=1}^{n} \frac{\partial}{\partial p_i} p_1 \frac{\partial}{\partial q_i} mq_1 - \frac{\partial}{\partial q_i} p_1 \frac{\partial}{\partial p_i} mq_1 \\ &= m \end{aligned}$$

Here $m \in C^\infty(X)$ is really the constant function equal to m at all points of X. But what vector field on X does this observable generate? What is $v_m = \{m, \cdot\}$? It is zero! It generates this flow:

$$\phi\colon \mathbb{R} \times X \to X$$

$$(t, x) \mapsto x$$

All constant functions, or indeed, all locally constant functions f on any Poisson manifold give $v_f = 0$.

picture of phase space with two connected components

The problem is that this diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\alpha"] \arrow[dr, "\gamma"'] & & \mathrm{Vect}(X) \\ & C^\infty(X) \arrow[ur, "\beta"'] & \end{tikzcd}$$

β is not 1-1; it sends all locally constant functions to 0, so we can have

$$\beta\gamma[x, y] = [\beta\gamma(x), \beta\gamma(y)]$$

even though

$$\gamma[x, y] \neq [\gamma(x), \gamma(y)].$$

This also means that different choices of γ can make this diagram commute.

Definition 2 *If G is a Lie group acting on a Poisson manifold X:*

$$A\colon G \times X \to X$$

we say A is **weakly** *Hamiltonian if there exists a linear map*

$$\gamma\colon \mathfrak{g} \to C^\infty(X)$$

such that

$$\alpha = \beta\gamma$$

with γ not necessarily a Lie algebra homomorphism.

In our example we have:

$$\{\gamma(x), \gamma(y)\} = \gamma[x, y] + c(x, y)$$

where $c(x, y)$ is a locally constant function, so $\{c(x, y), \cdot\} = 0$. In this situation we call c a "2-cocycle".

It turns out that weakly Hamiltonian actions of the Galilei group $G(n+1)$ on the Poisson manifold $T^*\mathbb{R}^n$ can completely classified: there is one for each $m \in \mathbb{R}$. This m specifies the cocycle ... but physically it is the **mass** of a particle in $\mathbb{R}^n$. So the concept of mass is inevitable ... even without gravity around!