Classical Mechanics, Lecture 2

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lecture by John Baez

notes by Alex Hoffnung

1 Introduction

There are two approaches to classical mechanics:

- *The Lagrangian approach*: treats position and velocity as fundamental and describes how they change in time if you are given the **Lagrangian**, a function of position and velocity.
- *The Hamiltonian approach*: treats position and momentum as fundamental and describes how they change in time if you are given the **Hamiltonian**, a function of position and momentum. What is it? It is the **energy** (e.g. kinetic energy + potential energy).

The Hamiltonian approach, which is what we are going to describe, leads to a lot of interesting mathematics. It turns out that giving the position and momentum of a particle specifies a point in a space called the 'phase space' or 'state space' of the particle. Mathematically this is often a 'cotangent bundle' — an important concept from the theory of manifolds. A cotangent bundle is an example of a 'symplectic manifold', which in turn is an example of a more general thing called a 'Poisson manifold'. These are the things we want to understand. But we will work our way up to these abstractions starting from the basics!

In our historical overview, I led up through the history of physics to Newton's great *Principia* and his three laws. Now we will actually start doing some physics à la Newton. We are going to start out by thinking about a classical particle moving in n-dimensional space. Think $n = 3$ if you like — but other dimensions are interesting too!

2 A classical particle in $\mathbf{R}^n$

In classical mechanics a particle traces out a path in some space, say $\mathbb{R}^n$:

$$q\colon \mathbb{R} \to \mathbb{R}^n$$

Here $\mathbb{R}$ stands for 'time' and $\mathbb{R}^n$ stands for 'space': the particle's position in space is a function of time. We say that $q(t)$ is the **position** of the particle at time t. We define the **velocity**

$$v(t) \;=\; \dot{q}(t) \;=\; \frac{dq(t)}{dt}$$

and **acceleration**

$$a(t) \;=\; \ddot{q}(t) \;=\; \frac{dv(t)}{dt}.$$

Any particle has a **mass** $m > 0$, and Newton's 2nd law says

$$F \;=\; ma,$$

i.e.

$$F(t) \;=\; m\ddot{q}(t)$$

for all $t \in \mathbb{R}$, where $F\colon \mathbb{R} \to \mathbb{R}^n$ is called the **force**. We will assume that q, F and indeed all functions we discuss are **smooth** or C^∞ — meaning they have infinitely many continuous derivatives. Then, with luck, we can solve this second order differential equation, namely Newton's second law:

$$\ddot{q}(t) \;=\; \frac{F(t)}{m}$$

for $q(t)$ if we know $F(t)$, m, $q(t_0)$, and also its time derivative $\dot{q}(t_0)$ for some t_0 (often called "time zero").

Examples:

1. *A free particle.* If $F(t) = 0$, we have a **free particle**. Then $\ddot{q}(t) = 0$ so $\dot{q}(t)$ is constant, say $v \in \mathbb{R}^n$, and
$$q(t) = q(0) + tv = q(0) + t\dot{q}(0).$$
We recover Newton's first law: a free particle moves along a line in $\mathbb{R}^n$ at constant velocity.

2. *A particle near the Earth's surface feeling only the force of gravity.* This force is approximately independent of time and position:
$$F(t) = (0, 0, -mg).$$
Here we are in $\mathbb{R}^3$ and g is the downwards acceleration due to gravity - approximately 9.8 meters/second^2.

   Homework 1: Solve Newton's second law $F = ma$ for $q(t) \in \mathbb{R}^3$ for this $F(t)$ - find $q(t)$ in terms of the initial position $q(0)$ and $\dot{q}(0)$. Hint: the path it traces out is a parabola!

3. *The* **simple harmonic oscillator**. Here n = 1:
$$picture\ of\ mass\ m\ on\ spring\ at\ equlibrium\ and\ picture\ of\ stretched\ spring$$
$$with\ height\ difference\ q(t)$$
caption: The equilibrium position has q = 0.
caption: The position q(t) is measured relative to the equilibrium position.
The force is approximately given by **Hooke's Law**:
$$F(t) \;=\; -\,kq(t),$$
where $k > 0$ is the **spring constant**. There is a famous fact that physicists often quote, which is that 'to first order everything is linear'. This is tautological, of course, but it really says that most functions are differentiable and we can approximate by the first derivative. So, it's not so absurd to think that approximately, at least, the force will depend in a linear way on position in many situations. That's why the harmonic oscillator is so important.

   Homework 2: Solve Newton's second law for $q(t) \in \mathbb{R}$ (when the force is given by Hooke's law) in terms of m, k, $q(0)$, $\dot{q}(0)$ and find the period P of the oscillation and the frequency ω, where $\omega = \dfrac{2\pi}{P}$ Hint: It oscillates!

3 Momentum and Energy

To go further, let's study two ways to integrate the force $F\colon \mathbb{R} \to \mathbb{R}^n$:
$$\int_{t_1}^{t_2} F(t)dt \quad \in \mathbb{R}^n$$
$$\int_{t_1}^{t_2} F(t) \cdot \dot{q}(t)dt \quad \in \mathbb{R}$$

We can do these assuming Newton's second law:

$$\int_{t_1}^{t_2} F(t) \; = \; \int_{t_1}^{t_2} m\ddot{q}(t)dt \; = \; m\dot{q}(t) \mid_{t_1}^{t_2} \; = \; p(t) \mid_{t_1}^{t_2},$$

where $p\colon \mathbb{R} \to \mathbb{R}^n$ is the particle's **momentum**:

$$p(t) \; = \; m\dot{q}(t).$$

So: the change in momentum is the integral of force. Or:

$$F(t) \; = \; \dot{p}(t).$$

Now let's do the other integral:

$$\int_{t_1}^{t_2} F(t) \cdot \dot{q}(t)dt \quad \in \mathbb{R} \; = \; \int_{t_1}^{t_2} m\ddot{q} \cdot \dot{q}(t)dt \; = \frac{1}{2} m\dot{q}(t) \cdot \dot{q}(t) \mid_{t_1}^{t_2} \; = \; T(t) \mid_{t_1}^{t_2},$$

where $T\colon \mathbb{R} \to \mathbb{R}$ is the particle's **kinetic energy**:

$$T(t) \; = \; \frac{1}{2} mv(t)^2.$$

So: the change in kinetic energy — the **work** — is the integral of $F(t) \cdot v(t)$. To understand this term 'work', imagine your job is to hold up a boulder all day long and a physicist comes along and asks you how much work you have done today. You would have to say you had done none! How unfair! But if you really think about it you could have done the same job with no work by just propping it up with another little rock.

4 Conservative Forces

Sometimes the force $F(t)$ only depends on the position and velocity of the particle:

$$F(t) \; = \; f(q(t), \dot{q}(t)).$$

Sometimes it only depends on position:

$$F(t) \; = \; f(q(t)),$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is some fixed (smooth) vector field. And sometimes, when we're really lucky,

$$f \; = \; -\nabla V$$

for some function $V\colon \mathbb{R}^n \to \mathbb{R}$. In this case we can do the integral for work another way:

$$\int_{t_1}^{t_2} F(t) \cdot \dot{q}(t)dt \; = \; -\int_{t_1}^{t_2} \nabla(q(t)) \cdot \dot{q}(t)dt \; = \; \int_{t_1}^{t_2} \frac{d}{dt} V(q(t))dt \; = \; -V(q(t)) \mid_{t_1}^{t_2}$$

So in this case:

$$\begin{aligned} T(t) \mid_{t_1}^{t_2} \; &= \; -V(q(t)) \mid_{t_1}^{t_2} \\ T(t_2) - T(t_1) \; &= \; V(q(t_1)) - V(q(t_2)) \\ T(t_2) + V(q(t_2)) \; &= \; T(t_1) + V(q(t_1)) \end{aligned}$$

So we call V the **potential energy** and $E = T + V$ the **energy**:

$$E(t) \; = \; T(t) + V(t)$$

We conclude that in this situation, energy is **conserved** (constant as a function of time). So: if $F(t) = -\nabla V(q(t))$ we call the force F **conservative**.