Classical Mechanics, Lecture 6

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1 Galilean Symmetry and its Conserved Quantity

Last time we discovered there was a symmetry called Galilean symmetry, but we did not know a corresponding conserved quantity. Given n particles in $\mathbb{R}^3$ interacting via central forces, if $q_i \colon \mathbb{R} \to \mathbb{R}^3$ is a solution of Newton's 2^{nd} law, we get a new solution

$$\tilde{q}_i(t) \;=\; q_i(t) + tv$$

where $v \in \mathbb{R}^3$. This is called **Galilean symmetry**; Galilean symmetries form a group, $\mathbb{R}^3$. What are the conserved quantities?
Our system of particles has a **total mass**:

$$m \;=\; \sum_{i=1}^{n} m_i$$

and a **center of mass**

$$q(t) \;=\; \frac{\sum m_i q_i(t)}{m}.$$

We have also discussed the **total momentum**

$$p(t) \;=\; \sum_{i=1}^{n} p_i(t)$$

which is also conserved. Note:

$$p(t) \;=\; m\dot{q}(t)$$

so the center of mass moves at a constant velocity, so:

$$q(t) \;+\; q(0) + tv$$

for some $v \in \mathbb{R}^3$. So

$$q(t) - tv \in \mathbb{R}^3$$

is a conserved quantity! This is "center of mass at time zero" - this is the conserved quantity corresponding to Galilean symmetry.

$$q(t) - tv \;=\; \frac{\sum m_i q_i(t)}{m} - \frac{t\sum m_i \dot{q}_i(t)}{m}.$$

Compare this to total momentum:

$$p(t) \;=\; \sum m_i \dot{q}_i(t).$$

Note: the center of mass at time zero has "explicit time dependence" - not just a function of $q_i(t)$ and $\dot{q}_i(t)$.

2 Hamilton's Equations

Let's just consider a single particle in $\mathbb{R}^n$, with position

$$q: \mathbb{R} \to \mathbb{R}^n$$

satisfying newton's 2^{nd} law:

$$m\ddot{q}_i(t) = \frac{\partial V}{\partial q_i}(q(t))$$

for some potential $V: \mathbb{R}^n \to \mathbb{R}$. This equation is 2^{nd}-order, so you an rewrite it as a pair of 1^{st}-order equations:

$$\begin{aligned} \dot{q}_i(t) &= \frac{1}{m} p_i(t) \quad (**) \\ \dot{p}_i(t) &= \frac{\partial V}{\partial q_i}(q(t)) \end{aligned}$$

describing the rate of position and momentum - these are "equal partners" in the Hamiltonian approach. The right-hand side is related to energy

$$\begin{aligned} E &= \frac{1}{2} m\dot{q}^2 + V(q) \\ &= \frac{p^2}{2m} + V(q) \end{aligned}$$

The **Hamiltonian** $H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the energy as a function of $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$:

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

Note:

$$\begin{aligned} \frac{\partial H}{\partial p_i}(q,p) &= \frac{p_i}{m} \\ \frac{\partial H}{\partial q_i}(q,p) &= \frac{\partial V}{\partial q_i} \end{aligned}$$

So, (**) are equivalent to Hamilton's equations:

$$\begin{aligned} \frac{d}{dt} q_i(t) &= \frac{\partial H}{\partial p_i}(q(t), p(t)) \\ \frac{d}{dt} p_i(t) &= -\frac{\partial H}{\partial q_i}(q(t), p(t)) \end{aligned}$$

This pattern reminds of us rotating by 90 degrees in the plane or multiplying by i. This is the secret expanation of what is going on!

3 Poisson Brackets

We call $\mathbb{R}^n$ the **phase space** of a particle in n-dimensions - a point in it specifies the particles position and momentum

$$(q,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We call any smooth function $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ an **observable**. We can ask how an observable "evolves in time" to give a new observable F_t, $(t \in \mathbb{R})$ - F measured after you wait a certain amount of time. Mathematically, $F_t: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the observable:

$$F_t(qp) \;=\; F(q(t), p(t))$$

where $q(t)$, $p(t)$ are the solution of Hamilton's equations with $q(0) = q$, $p(0) = p$.
How does F_t change as time passes:

$$\frac{d}{dt} F_t \;=\; ?$$

Calculate

$$\begin{aligned}
\left(\frac{d}{dt} F_t\right)(q, p) &= \frac{d}{dt} F(q(t), p(t)) \\
&= \sum_i \frac{\partial F}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial F}{\partial p_i} \frac{dp_i}{dt} \\
&= \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i}
\end{aligned}$$

For this reason we invent **Poisson brackets**: given any pair of observables $F, G: \mathbb{R}^{2n} \to \mathbb{R}$, we let

$$\{F, G\} \;=\; \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i}$$

In this notation Hamilton's equations say:

$$\begin{aligned}
\frac{d}{dt} F_t(q, p) &= \{H, F\}(q(t), p(t)) \\
&= \{H, F\}_t(q, p)
\end{aligned}$$

or:

$$\frac{d}{dt} F_t \;=\; \{H, F\}_t.$$

We'll say "the Hamiltonian **generates** time evolution". In fact, other interesting observables generate other interesting symmetries.
Consider spatial translation:

$$q \mapsto q + sk, \quad k \in \mathbb{R}^n$$
$$p \mapsto p, \quad s \in \mathbb{R}$$

We could look at how an observable changes under spatial translation, define:

$$F_s(q, p) \;=\; F(q + sk, p)$$

and compute

$$\begin{aligned}
\frac{dF_s}{ds}(q, p) &= \frac{d}{ds} F(q + sk, p) \\
&= \sum_i \frac{\partial F}{\partial q_i} k_i \\
&= \{p \cdot k, F\}
\end{aligned}$$

where $p \cdot k$ is "momentum in the k direction". So: "translations in the k direction are generated by momentum in the k direction."