Classical Mechanics, Lecture 7

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1 The Poisson Bracket

If we have one particle in $\mathbb{R}^n$, we call $\mathbb{R}^n$ the **configuration space** - the space of possible positions of the whole system. If we have n particles in $\mathbb{R}^3$ the configuration space is $\mathbb{R}^{3n}$, since a point in here is an n-tuple $(q_1, \ldots, q_n)$ where $q_i \in \mathbb{R}^3$ is the position of the i^{th} particle.

picture of a pendulum swinging around a pivot

For a pendulum that can swing all the way around in a plane, the configuration space is S^1 - the circle. For 42 pendula, the configuration space is

$$S^1 \times \cdots \times S^1 = T^{42}$$

(the 42-dimensional torus). The configuration space of a rigid body in $\mathbb{R}^3$ with center of mass at $0 \in \mathbb{R}^3$ is

$$\begin{aligned} \mathrm{SO}(3) &= \{3 \times 3 \text{ rotation matrices}\} \\ &= \{R\colon \mathbb{R}^3 \to \mathbb{R}^3 \text{ such that } RR^T = 1, \det R = 1\} \end{aligned}$$

a 3-dim manifold. The configuration space of a rigid body in $\mathbb{R}^3$ is $\mathrm{SO}(3) \times \mathbb{R}^3$, where a point in $\mathbb{R}^3$ specifies the center of mass.

But Hamiltonian mechanics focuses not on the configuration space but on the **phase space** or **state space**, where a point specifies the position and momentum of the system. For a single particle in $\mathbb{R}^n$, the phase space is:

$$\mathbb{R}^{2n} \cong \mathbb{R}^n \times \mathbb{R}^n \in (q, p)$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. For n particles in $\mathbb{R}^3$, the phase space is

$$\mathbb{R}^{6n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n}.$$

For a rigid body in $\mathbb{R}^3$ with center of mass at the origin, the phase space is $\mathrm{SO}(3) \times \mathbb{R}^3$ where $q \in \mathrm{SO}(3)$ is the position and $p \in \mathbb{R}^3$ is actually the **angular momentum**, usually called J. Actually a better description is $T^*\mathrm{SO}(3)$ - the cotangent bundle of $\mathrm{SO}(3)$.

In our example of a particle in $\mathbb{R}^n$, we said an **observable** is a smooth function

$$F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

by

$$(q, p) \mapsto F(q, p)$$

from the phase space to $\mathbb{R}$, sending each point of phase space (or **state** of our system) to the value of the observables.

If X is any manifold, thought of as a phase space, let

$$C^\infty(X) = \{\text{smooth functions } F\colon X \to \mathbb{R}\}$$

be the set of observables. This is a commutative algebra.

Definition 1 *A* **commutative algebra** *is a (real) vector space A equipped with a product satisfying:*

1. **bilinearity***:*

$$(\alpha F + \beta G)H = \alpha FH + \beta GH, \quad \forall \alpha, \beta \in \mathbb{R}$$
$$F(\alpha G + \beta H) = \alpha FG + \beta FH, \quad F, G, H \in A$$

2. **commutativity***:*

$$FG = GF$$

3. **associativity***:*

$$(FG)H = F(GH)$$

$C^\infty(X)$ becomes a commutative algebra with the "obvious" addition, scalar multiplication, and multiplication:

$$(\alpha F)(x) = \alpha F(x), \quad x \in X$$
$$(F+G)(x) = F(x) + G(x)$$
$$(FG)(x) = F(x)G(x)$$

But, in our example $X = \mathbb{R}^{2n}$, there's another operation, the Poisson bracket:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}$$

This makes $C^\infty(X)$ into a 'Lie algebra':

Definition 2 *A* **Lie algebra** *is a (real) vector space A equipped with a "Lie bracket" $\{\cdot, \cdot\}$*

1. **bilinearity***:*

$$\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$$
$$\{F, \alpha G + \beta H\} = \alpha\{F, G\} + \beta\{G, H\}$$

2. **antisymmetry***:*

$$\{F, G\} = -\{G, F\}$$

3. **Jacobi identity***:*

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$$

(Rule 3 looks like the product rule: $dGH = (dG)H + g(dH)$.)

In fact for all classical mechanics problems, the algebra of observables $C^\infty(X)$ is always both a commutative algebra and a Lie algebra, but even better, they fit together to form a **Poisson algebra**.

Definition 3 *A* **Poisson algebra** *is a vector space A with a product making it into a commutative algebra and a bracket $\{\cdot, \cdot\}$ making it into a Lie algebra such that*

$$\{F, GH\} = \{F, G\}H + G\{F, H\}$$

We saw that for a particle in $\mathbb{R}^n$, with energy given by:

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

where $V: \mathbb{R}^n \to \mathbb{R}$, Newton's 2^{nd} law can be rewritten as Hamilton's equations:

$$\frac{d}{dt} q_i(t) = \frac{\partial H}{\partial p_i}(q(t), p(t))$$

$$\frac{d}{dt} p_i(t) = -\frac{\partial H}{\partial q_i}(q(t), p(t))$$

If $H \in C^\infty(\mathbb{R}^{2n})$ is nice, these have a unique smooth solution for any choice of initial $q(0) = q$ and $p(0) = p$. Then we get a function

$$\phi: \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

by

$$(t, q, p) \mapsto (q(t), p(t))$$

which describes time evolution. Often we write

$$\phi(t, q, p) = \phi_t(q, p)$$

where

$$\phi_t: \mathbb{R}^{2n} \to \mathbb{R}^{2n}.$$

Then we can say how any observable changes with time: given $F \in C^\infty(\mathbb{R}^{2n})$ we get

$$F_t(q, p) = F\phi_t(q, p)$$

and Hamilton's equations say:

$$\begin{aligned} \frac{d}{dt} F_t &= \sum_{i=1}^{n} \frac{\partial F}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial F}{\partial p_i} \frac{dp_i}{dt} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \\ &= \{H, F\}_t \end{aligned}$$

This is why the Poisson algebra is important.