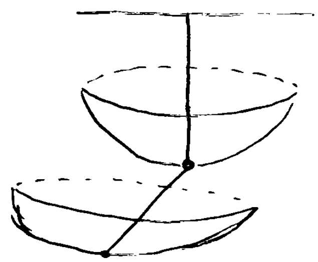
The Euler-Lagrange Equation

4 April 2005

Now let's consider an arbitrary classical system whose space of "positions" is some manifold Q — the configuration space.

For example: a spherical double pendulum



$$Q = S^2 \times S^2$$

We'll think of our system as "a particle in Q." As time passes, it traces out a path

$$q : [t_0, t_1] \longrightarrow Q$$

& its velocity will be $\dot{q}(t) \in T_{q(t)}Q$ — a tangent vector.

Let $\mathcal{P}$ be the space of smooth paths from $a \in Q$ to $b \in Q$:

$$\mathcal{P} = \{ q : [t_0, t_1] \to Q \; : \; q(t_0) = a, \; q(t_1) = b \}$$

(really an ∞-dim manifold, but we won't discuss this now.)

We'll let the Lagrangian L be any smooth function of position and velocity (note: not explicitly of time, for simplicity)

$$L : TQ \longrightarrow \mathbb{R}$$

and we define the action

$$S : \mathcal{P} \longrightarrow \mathbb{R}$$

by

$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, dt$$

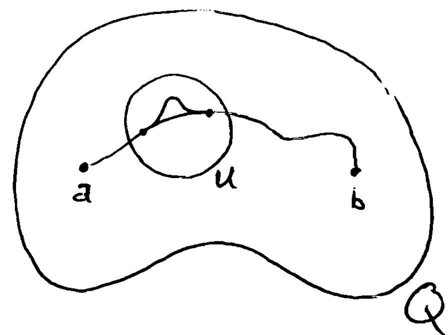
The path(s) the particle will actually take is a critical point of S. In other words, a path $q \in \mathcal{P}$ such that for any smooth 1-parameter family of paths $q_s \in \mathcal{P}$ with $q_0 = q$, we have

$$\frac{d}{ds} S(q_s) \Big|_{s=0} = 0$$

We write $\frac{d}{ds}\Big|_{s=0}$ as "δ", so

$$\delta S = 0$$

What does this really amount to? For now, let's pick coordinates in a nbhd U of some point $q(t) \in Q$ and only consider variations q_s s.t. $q_s = q$ outside U. Then we can restrict attention to a subinterval $[t_0', t_1'] \subseteq [t_0, t_1]$ s.t. $q_s(t) \in U$ for $t_0' \leq t \leq t_1'$. Let's rename t_0' & t_1' "t_0 & t_1".



So: we can now use coordinates on U:

$$\varphi : U \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \varphi(x) = (x^1, \ldots, x^n)$$

& we also have coordinates

$$d\varphi : TU \longrightarrow T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$
$$(x,y) \longmapsto d\varphi(x,y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$$

where $y \in T_xQ$. We restrict $L : TM \to \mathbb{R}$ to $TU \subseteq TM$

and then describe it using the coordinates x^i, y^i on TU.
x^i are <u>position</u> coordinates, y^i are <u>velocity</u> coordinates.
Using these coordinates, we get:

$$\begin{aligned}\delta S &= \delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt \\ &= \int_{t_0}^{t_1} \delta L(q(t), \dot{q}(t))\, dt \\ &= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}\, \delta q^i(t) + \frac{\partial L}{\partial y^i}\, \delta \dot{q}^i(t)\end{aligned}$$

(with summation over i implied according ~~tha~~ the Einstein summation convention)

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \right) \delta q^i(t)\, dt$$

note: boundary terms vanish since we choose variations that vanish at endpts:

Since this should vanish for all δq, we need

$$\frac{d}{dt} \frac{\partial L}{\partial y^i} = \frac{\partial L}{\partial x^i}$$

This is <u>necessary</u> to get $\delta S = 0 \ \forall \delta q$, but in fact it's also sufficient. Physicists always call the coordinates x^i, y^i on TU "q^i" & "$\dot{q}^i$", despite the fact that these also have another meaning, namely the x^i & y^i coords of $(q(t), \dot{q}(t)) \in TU$. So physicists write

$$\boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}}$$

& these are called the <u>Euler-Lagrange equations</u>.

To understand these, compare our favorite example

$$Q = \mathbb{R}^n$$

$$L(q, \dot q) = \tfrac{1}{2} m \dot q \cdot \dot q - V(q)$$

$$\tfrac{1}{2} m \dot q^i q_i$$

TERMS	MEANING IN OUR EXAMPLE	MEANING IN GENERAL
$\frac{\partial L}{\partial \dot q^i}$	$m\dot q_i$	the <u>momentum</u> p_i
$\frac{\partial L}{\partial q^i}$	$-(\nabla V)_i$	the <u>force</u> F_i

So we get general concepts of momentum & force, & E-L eq^ns say

$$\dot p = F$$

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Noether's Theorem

This gives conserved quantities from symmetries & in particular conservation of energy from time translation invariance. To handle time translations, we need to replace our paths $q : [t_0, t_1] \to Q$ by paths $q : \mathbb{R} \to Q$, & define a new space of paths

$$\mathcal{P} = \{ q : \mathbb{R} \to Q \}$$

The bad news is that the action

$$S(q) = \int_{-\infty}^{\infty} L(q(t), \dot{q}(t))\, dt$$

typically won't converge — so S is no longer a function of the space of paths. Nonetheless, if $\delta q = 0$ outside some finite interval

$$\text{"}\delta S\text{"} := \int_{-\infty}^{\infty} \frac{d}{ds} L(q_s(t), \dot{q}_s(t)) \Big|_{s=0} dt$$

<u>will</u> converge, since the integrand is smooth & vanishes outside this interval. Moreover, demanding that this δS vanishes for all such variations δq is enough to imply the Euler-Lagrange equations:

$$\delta S = \int_{-\infty}^{\infty} \frac{d}{ds} L(q_s(t), \dot{q}_s(t)) \Big|_{s=0} dt$$
$$= \int_{-\infty}^{\infty} \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \, dt$$
$$= \int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt$$

(no boundary terms since $\delta q = 0$ near $t = \pm\infty$) & this vanishes for all compactly supported smooth δq iff

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Recall :

$\frac{\partial L}{\partial \dot{q}_i} = p_i$ is <u>momentum</u> by definition

$\frac{\partial L}{\partial q_i} = \dot{p}_i$ is force (by Euler Lagrange eqn.)

(note the similarity to Hamilton's eqns — change L to H and stick in a minus sign.)

Now for Noether's theorem :

<u>Def</u> — A <u>one-parameter family of symmetries</u> of a Lagrangian system $L : TQ \to \mathbb{R}$ is a smooth map

$$F : \mathbb{R} \times \mathcal{P} \longrightarrow \mathcal{P} \qquad \text{with } q_0 = q$$
$$(s, q) \longmapsto q_s$$

such that

$$\delta L = \frac{d}{dt} \ell$$

for some $\ell : TQ \to \mathbb{R}$, i.e.

$$\frac{d}{ds} L(q_s(t), \dot{q}_s(t))\Big|_{s=0} = \frac{d}{dt} \ell(q(t), \dot{q}(t))$$

for all paths q.

Remark: the simplest case is $\delta L = 0$; then we really have a way of moving paths around $(q \mapsto q_s)$ that doesn't change the Lagrangian — i.e. a symmetry of L in the obvious sense. $\delta L = \frac{d}{dt}\ell$ is a sneaky generalization whose usefulness will become clear.

Noether's Theorem — Suppose F is a one-parameter family of symmetries of the Lagrangian system $L: TQ \to \mathbb{R}$. Then

$$p^i \delta q_i - \ell$$

is conserved, i.e. its time derivative is zero for any path $q \in \mathcal{P}$ satisfying the Euler-Lagrange eqns. (i.e.

$$\frac{d}{dt}\left[\frac{\partial L}{\partial y^i}(q(s), \dot{q}(s)) \frac{d}{ds} q_s^i(t)\Big|_{s=0} - \ell(q(t), \dot{q}(t))\right] = 0$$

if we write it out in gory detail).

Proof:
$$\frac{d}{dt}(p_i \delta q^i - \ell) = \dot{p}_i \delta q^i + p_i \delta \dot{q}^i - \frac{d}{dt}\ell$$
$$= \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q} - \delta L$$
$$= \delta L - \delta L = 0. \quad \blacksquare$$

Examples:

1) Conservation of energy (most important example!)

<u>All</u> our Lagrangian Systems have time translation invariance: we have the one-parameter family of symmetries

$$q_s(t) = q(t+s)$$

This indeed gives

$$\delta L = \dot{L}$$

So here we take $\ell = L$! We get the conserved quantity

$$p_i \delta q^i - \ell = p_i \dot{q}^i - L$$

which we call the <u>energy</u>. E.g. if $Q = \mathbb{R}^n$ &

$$L = \frac{1}{2} m \dot{q}^2 - V(q)$$

then this quantity is

$$m\dot{q}\cdot\dot{q} - \underbrace{\left(\frac{1}{2} m\dot{q}\cdot\dot{q} - V\right)}_{K - V} = \underbrace{\frac{1}{2} m\dot{q}^2 + V(q)}_{K + V}$$

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Conserved Quantities From Symmetries

We've seen that any 1-parameter family

$$F_s : \mathcal{P} \longrightarrow \mathcal{P}$$
$$q \longmapsto q_s$$

which satisfies

$$\delta L = \dot{\ell}$$

for some function $\ell = \ell(q, \dot{q})$ (where $\delta L := \frac{d}{ds} L(q_s(t), \dot{q}_s(t))\big|_{s=0}$)

gives a conserved quantity

$$p_i \delta q^i - \ell$$

1) Time translation symmetry: for any Lagrangian system $L : TQ \to \mathbb{R}$ we have a 1-param family of symmetries

$$q_s(t) = q(t+s)$$

since

$$\delta L = \dot{L}$$

so we get a conserved quantity called __energy__ or the __Hamiltonian__

$$H = p_i \dot{q}^i - L$$

For example: a particle on $\mathbb{R}^n$ in a potential V has $Q = \mathbb{R}^n$, $L(q,\dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q)$. This has

$$K \quad - V$$

$$p_i \dot{q}^i = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i = m\dot{q}^2 = 2K$$

so

$$H = p_i \dot{q}^i - L = 2K - (K - V) = K + V$$

as we'd hope.

2) For a free particle in $\mathbb{R}^n$, we have $Q = \mathbb{R}^n$ & $L = K = \frac{1}{2}m\dot{q}^2$. This has <u>spatial translation</u> symmetries: for any $v \in \mathbb{R}^n$ we have the symmetry

$$q_s(t) = q(t) + sv$$

with

$$\delta L = 0$$

since $\delta\dot{q} = 0$ & L depends only on $\dot{q}$, not q. (Since L doesn't depend on q_i, we call q_i an <u>ignorable coordinate</u>; as above, these always give symmetries, hence conserved quantities. It is often useful to change coordinates so as to make some of them ignorable!) Here we get a conserved quantity called <u>momentum in the v direction</u>:

$$p_i \delta q^i = m\dot{q}_i v^i = m\dot{q}\cdot v.$$

note: subtle difference between two uses of 'momentum': a conserved quantity gen. by translation & the momentum $\frac{\partial L}{\partial \dot{q}_i}$ conjugate to p_i — they happen to be the same here!

Since this is conserved for all v we say $m\dot{q} \in \mathbb{R}^n$ is conserved.

(In fact, the whole Lie group $G = \mathbb{R}^n$ is acting as translation symmetries and we're getting a $\mathfrak{g}\,(= \mathbb{R}^n)$-valued conserved quantity!)

3) The free particle in $\mathbb{R}^n$ also has <u>rotation</u> symmetry. Consider any $X \in so(n)$ (a skew-symmetric $n \times n$ matrix); then for all $s \in \mathbb{R}$ the matrix e^{sX} is in $SO(n)$, i.e. describes a rotation. This gives a 1-parameter family of symmetries

$$q_s(t) = e^{sX} q(t)$$

which has

$$\delta L = \underbrace{\frac{\partial L}{\partial q^i}}_{\substack{\| \\ 0 \\ q_i \text{ ignorable}}} \delta q^i + \underbrace{\frac{\partial L}{\partial \dot{q}^i}}_{\substack{\| \\ p_i}} \delta \dot{q}^i = m\dot{q}_i \, \delta \dot{q}^i$$

and

$$\begin{aligned} \delta \dot{q}^i &= \frac{d}{ds} \dot{q}^i_s \Big|_{s=0} \\ &= \frac{d}{ds} \frac{d}{dt} (e^{sX} q) \Big|_{s=0} \\ &= \frac{d}{dt} Xq \\ &= X\dot{q} \end{aligned}$$

Note: this whole calculation just says the kinetic energy doesn't change when the velocity is <u>rotated</u> (without changing the magnitude).

So

$$\begin{aligned} \delta L &= m\dot{q}_i X^i_j \dot{q}^j \\ &= m \, \dot{q} \cdot (X\dot{q}) = 0 \end{aligned}$$

since X is skew symmetric. So we get a conserved quantity, the <u>angular momentum</u> in the X direction.

$$p_i \delta q^i = m\dot{q}_i \cdot (Xq)^i$$ ($\delta q^i = Xq$ just as $\delta \dot{q}^i = X\dot{q}$ in our previous calculation)

or if X has zero entries except in ij & ji positions, where it's ± 1, we get

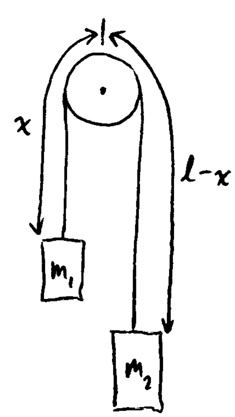
$$m(\dot{q}_i q^j - \dot{q}_j q^i)$$

the "ij component of angular momentum." If $n=3$, we write these as $m\dot{q} \times q$.

Example Problems

To see how this formalism functions, let's do some problems! It's vastly superior to the naive $F=ma$ formalism, since it allows the config space to be any manifold & allows us to easily use any coordinates we wish.

1) Atwood machine:



A frictionless pulley with two masses m_1 & m_2 hanging from it.

$$K = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$V = -m_1 g x - m_2 g(l-x)$$