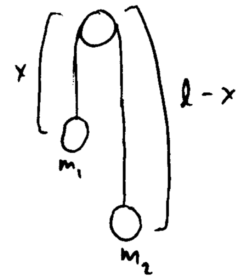
11 April 2005

Atwood Machine



$$K = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 (\dot{l-x})^2 = \tfrac{1}{2}(m_1+m_2)\dot{x}^2$$

$$V = -m_1 g x - mg(l-x)$$

$$L = K - V = \tfrac{1}{2}(m_1+m_2)\dot{x}^2 + m_1 g x + m_2 g(l-x)$$

Note: config space is $Q = (0, l) \ni x$

and $TQ = (0,l) \times \mathbb{R} \ni (x, \dot{x})$

& $L : TQ \to \mathbb{R}$. Note solns of E-L eqns will only be defined for <u>some</u> times $t \in \mathbb{R}$ — eventually the solution reaches the "edge" of Q.

Momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

Force is

$$F = \frac{\partial L}{\partial x} = (m_1 - m_2) g$$

The E-L eqns say

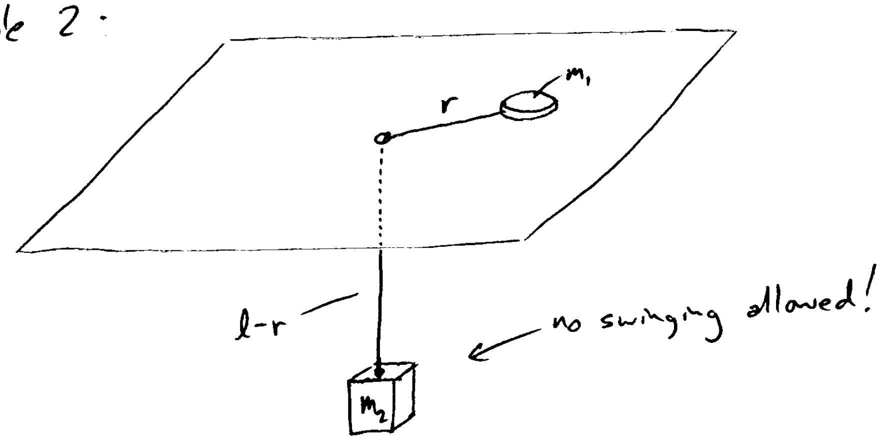
$$\dot{p} = F$$

$$(m_1+m_2)\ddot{x} = (m_1 - m_2) g$$

$$\ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g$$

So this is like a falling object in a downwards gravitational acceleration $\left(\frac{m_1-m_2}{m_1+m_2}\right)g$. Note this is 0 when $m_1 = m_2$, g if $m_2 = 0$.

Example 2:



Here Q = open disk of radius l, minus its center

$$= (0,l) \times S^1 \ni (r,\theta)$$

$$TQ = (0,l)\times S^1 \times \mathbb{R}\times\mathbb{R} \ni (r,\theta,\dot r,\dot\theta)$$

$$K = \tfrac{1}{2} m_1 (\dot r^2 + r^2\dot\theta^2) + \tfrac{1}{2} m_2 (\dot{l-r})^2$$

$$V = g m_2 (r-l)$$

$$L = \tfrac{1}{2} m_1 (\dot r^2 + r^2 \dot\theta^2) + \tfrac{1}{2} m_2 \dot r^2 + g m_2 (l-r)$$

We get moment

$$p_r = \frac{\partial L}{\partial \dot r} = (m_1+m_2)\dot r$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m_1 r^2 \dot\theta$$

Note: θ is an "ignorable coordinate" — θ doesn't appear in L — so there's a symmetry — rotational symmetry — and p_θ, the conjugate momentum, is conserved!

We have forces:

$$F_r = \frac{\partial L}{\partial r} = \underbrace{m_1 r \dot{\theta}^2}_{\text{centrifugal force pushes } m_1 \text{ out}} \underbrace{- g m_2}_{\text{gravity pulls } m_2 \text{ down \& thus pulls } m_1 \text{ in}}$$

$$F_\theta = \frac{\partial L}{\partial \theta} = 0 \quad - \quad \theta \text{ is ignorable.}$$

So: we get E-L eqns:

$$\dot{p}_r = F_r \qquad (m_1+m_2)\ddot{r} = m_1 r \dot{\theta}^2 - m_2 g$$

$$\dot{p}_\theta = 0 \qquad p_\theta = m_1 r^2 \dot{\theta} = J, \text{ a constant.}$$

Let's use our conservation law to eliminate $\dot{\theta}$ from the first eqn.:

$$\dot{\theta} = \frac{J}{m_1 r^2}$$

So

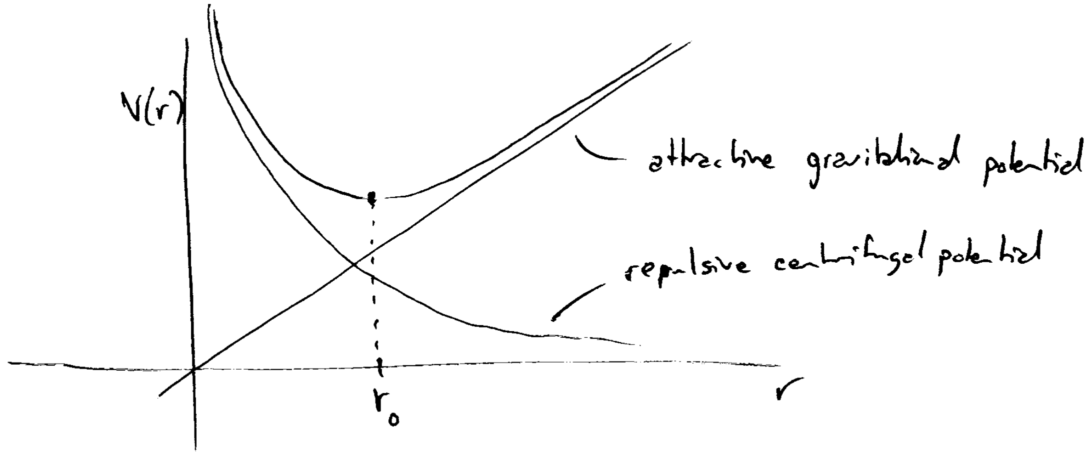
$$(m_1+m_2)\ddot{r} = \frac{J^2}{m_1 r^3} - m_2 g$$

So effectively we have a particle on $(0, \ell)$ of mass m_1+m_2 feeling a force

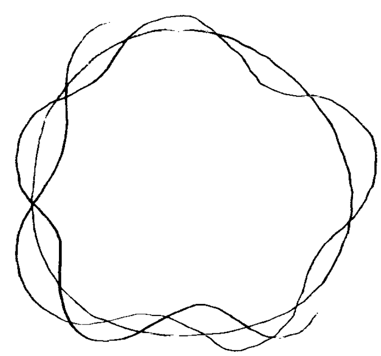
$$F_r = \frac{J^2}{m_1 r^3} - m_2 g$$

which could come from an "effective potential" $V(r)$ s.t. $V' = -F_r$.
So integrate $-F_r$:

$$V(r) = \frac{J^2}{2m_1 r^2} + m_2 g r$$



At r_0, the minimum of V, our mass m, will be in a stable circular orbit of radius r_0 (which depends on J). Otherwise we get orbits like:



3) A free particle in special relativity. Here the ~~parameter~~ coordinate parametrizing the particle's path in Minkowski spacetime need not be 'the' "time coordinate" since indeed there are many allowed time coordinates.

Minkowski spacetime is

$$\mathbb{R}^{n+1} \ni (\underbrace{x^0}_{\text{"time"}}, \underbrace{x^1, \dots, x^n}_{\text{"space"}})$$

if space is n-dimensional. This has a Lorentzian metric:

$$\begin{aligned} g(v,w) &= v^0 w^0 - v^1 w^1 - \dots - v^n w^n \\ &= \eta_{\mu\nu} v^\mu w^\nu \end{aligned}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & \ddots \\ & & & & & -1 \end{pmatrix}$$

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In special relativity we should take <u>spacetime</u> to be the configuration space of the point particle, so let Q be Minkowski spacetime (i.e. $\mathbb{R}^{n+1} \ni (x^0, \dots, x^n)$ with its Minkowski metric $\eta_{ij} = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & \ddots \, -1 \end{pmatrix}$.) Then the path of the particle is

$$\begin{aligned} q : \; &\mathbb{R} \longrightarrow Q \\ &\ \Psi \\ &\ t \end{aligned}$$

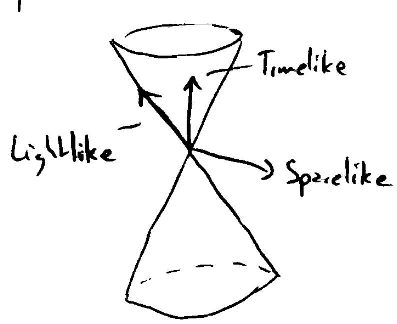
where t is a completely arbitrary parameter for the path – not necessarily x^0, and not necessarily "proper time".
Want some Lagrangian $L: TQ \to \mathbb{R}$, i.e. $L(q^i, \dot{q}^i)$ such that E-L eqns say our (free) particle moves at a constant velocity. Many Lagrangians do this, but the "best" one(s) give an action

$$S(q) = \int_{t_0}^{t_1} L(q^i(t), \dot{q}^i(t))\, dt$$

(for $q: [t_0, t_1] \to Q$) that's <u>independent</u> of the parameterization of the path – since the parameterization is "unphysical" (can't be measured). The obvious candidate for S is (mass times) arclength:

$$S = m \int_{t_0}^{t_1} \sqrt{\eta_{ij}\, \dot{q}^i(t)\, \dot{q}^j(t)}\; dt$$

or rather its Minkowski analogue, called <u>proper time</u> at least when $\dot{q}$ is a timelike vector, i.e. $\eta_{ij} \dot{q}^i \dot{q}^j > 0$, which says $\dot{q}$ points into the future (or past) lightcone:



and makes S <u>real</u>, in fact the time ticked off by a clock moving along the path $q : [t_0, t_1] \to \mathbb{R}$.

So let's take

$$L = \sqrt{\eta_{ij} \dot{q}^i \dot{q}^j}$$

& work out E-L eqns:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial}{\partial \dot{q}^i} m \sqrt{\eta_{ij} \dot{q}^i \dot{q}^j}$$

$$= m \frac{2 \eta_{ij} \dot{q}^j}{2\sqrt{\eta_{ij} \dot{q}^i \dot{q}^j}}$$

$$= m \frac{\eta_{ij} \dot{q}^j}{\sqrt{\eta_{ij} \dot{q}^i \dot{q}^j}} = \frac{m \dot{q}_i}{\|\dot{q}\|}$$

note: mass times "4-velocity." but note we are in n+1-dim spacetime

· Note this doesn't change when we change the parameter to accomplish $\dot{q} \longmapsto \alpha \dot{q}$. E-L eqs say

$$\dot{p}_i = F_i = \frac{\partial L}{\partial q^i} = 0$$

The meaning of this becomes clearer if we use "proper time" as our parameter (like parameterizing a curve by arclength) so that

$$\int_{t_0}^{t_1} \|\dot{q}\| \, dt = t_1 - t_0 \quad \forall t_0, t_1$$

(which fixes the parameterization up to an additive constant)

This implies $\|\dot{q}\| = 1$, so that

$$p_i = m \frac{\dot{q}_i}{\|\dot{q}\|} = m\dot{q}_i$$

and E-L eqns say

$$\dot{p}_i = 0 \implies m\ddot{q}_i = 0$$

so our particle moves unaccelerated along a straight line.

Note: this Lagrangian system has lots of symmetries coming from reparameterizing the path, so Noether's theorem gives lots of conserved quantities. (This is called the problem of time in GR — here we see it start to show up in SR.)

These symmetries work as follows: consider any (smooth) 1-parameter family of reparameterizations, i.e. diffeomorphisms

$$f_s : \mathbb{R} \longrightarrow \mathbb{R}$$

with $f_0 = 1_{\mathbb{R}}$. These act on the space of paths

$$\mathcal{P} = \{q : \mathbb{R} \to Q\}$$

as follows: given any $q \in \mathcal{P}$ we get

$$q_s(t) = q(f_s(t))$$

Note q_s is "physically indistinguishable" from q. Let's show that

$$\delta L = \dot{\ell} \qquad \text{(when E-L eqs. hold)}$$

so that Noether's thm. gives a conserved quantity

$$p_i \, \delta q^i - \ell$$

Here we go:

$$\delta L = \underbrace{\frac{\partial L}{\partial q^i}}_{\substack{\shortparallel \\ 0}} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i$$

$$= p_i \, \delta \dot{q}^i$$

$$= \frac{m \dot{q}_i}{\|\dot{q}\|} \frac{d}{ds} \dot{q}^i(f_s(t)) \Big|_{s=0}$$

$$= \frac{m \dot{q}_i}{\|\dot{q}\|} \frac{d}{dt} \frac{d}{ds} q^i(f_s(t)) \Big|_{s=0}$$

$$= \frac{m \dot{q}_i}{\|\dot{q}\|} \frac{d}{dt} \dot{q}^i(f_s(t)) \frac{d f_s(t)}{ds} \Big|_{s=0}$$

$$= \frac{m \dot{q}_i}{\|\dot{q}\|} \frac{d}{dt} \left(\dot{q}^i(t) \, \delta f \right)$$

$$= \frac{d}{dt} \left(p_i \dot{q}^i \, \delta f \right)$$

where in the last step we used E-L eqns. i.e. $\frac{d}{dt} p_i = 0$

So $\delta L = \dot{\ell}$ where $\ell = p_i \dot{q}^i \delta f$

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Last time: we saw the free relativistic particle has:

$$L = m\|\dot{q}\| = m\sqrt{\eta_{ij}\dot{q}^i\dot{q}^j}$$

& we considered reparameterization symmetries

$$q_s(t) = q(f_s(t)) \qquad f_s : \mathbb{R} \to \mathbb{R}$$

Note:

$$\delta q^i := \frac{d}{ds} q^i(f_s(t))\Big|_{s=0} = \dot{q}^i \delta f$$

So

$$\delta L = \underbrace{\frac{\partial L}{\partial q^i}}_{=0} \delta q^i + \underbrace{\frac{\partial L}{\partial \dot{q}^i}}_{=p} \delta \dot{q}^i$$

$$\begin{aligned} &= p_i \delta\dot{q}^i \\ &= p_i \frac{d}{dt} \delta q^i \\ &= p_i \frac{d}{dt} \dot{q}^i \delta f \\ &= \frac{d}{dt} \underbrace{p_i \dot{q}^i \delta f}_{\ell} \end{aligned}$$

So Noether's theorem gives a conserved quantity

$$p_i \delta q^i - \ell = p_i \dot{q}^i \delta f - p_i \dot{q}^i \delta f$$
$$= 0$$

So these conserved quantities VANISH! In short, we're seeing an example of what physicists call gauge symmetries gauge symmetries:

1) These are symmetries that permute different mathematized descriptions of the same physical situation — in this case, reparameterizations of a path.

2) These symmetries make it impossible to compute $q(t)$ given $q(0), \dot{q}(0)$: if $q(t)$ is a solution, so is $q(f(t))$ for any reparameterization $f: \mathbb{R} \to \mathbb{R}$. Highly nonunique soln. of E-L eqns!

3) These symmetries give conserved quantities that work out to equal zero!

Note: 1) is a subjective criterion, 2) & 3) are objective, and 3) is easy to test, so we often use it to distinguish gauge symmetries from physical symmetries.

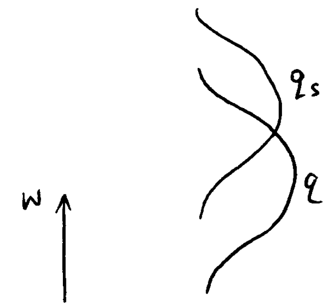
What's the Hamiltonian in this theory? Since it comes via Noether's thm. from time translation

$$q_s(t) = q(t+s)$$

& this is an example of a reparameterization (with $\delta f = 1$), we see from above that the Hamiltonian is zero! But there is another conserved quantity deserving the title of energy which is not zero, coming from the symmetry:

$$q_s(t) = q(t) + sw$$

where $w \in \mathbb{R}^{n+1}$ points in some timelike direction:



In fact, any vector w gives a conserved quantity:

$$\delta L = \underset{\overset{\parallel}{0}}{\frac{\partial L}{\partial q^i}} \delta q^i + \underset{\overset{\parallel}{p_i}}{\frac{\partial L}{\partial \dot{q}^i}} \delta \dot{q}^i$$

$$= p_i \delta \dot{q}^i$$

$$= p_i 0 = 0$$

since $\delta q^i = w^i$, $\delta \dot{q}^i = \dot{w}^i = 0$. This is $\dot{\ell}$ with $\ell = 0$, so Noether's thm says we get a conserved quantity:

$$p_i \delta q^i - \ell = p_i w^i$$

namely, momentum in the w direction. We know $\dot{p} = 0$ from E-L eqs, but here we see it coming from spacetime translation symmetry:

$$p = (\underbrace{p_0}_{\text{energy}}, \underbrace{p_1, \ldots, p_n}_{\text{spatial momentum}})$$

Relativistic Particle in an Electromagnetic Field

The electromagnetic field is described by a 1-form A on spacetime – the vector potential – such that

$$dA = F$$

is a 2-form containing the electric & magnetic fields

$$F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

& in 4-d spacetime

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

where E is the electric field & B is the magnetic field.
The action for a particle of charge e is

$$S = \underbrace{m\int_{t_0}^{t_1} \|\dot{q}\|\, dt}_{\text{proper time}} + \underbrace{e\int_q A}_{\text{the integral of } A \text{ along the path } q.}$$