18 April 2005

Given the Lagrangian for a particle with mass m & charge e in an electromagnetic vector potential A:

$$L(q,\dot{q}) = m\|\dot{q}\| + eA_i\dot{q}^i$$

let's work out the E-L equations:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m\frac{\dot{q}_i}{\|\dot{q}\|} + eA_i$$

$$= mv_i + eA_i$$

where $v \in \mathbb{R}^{n+1}$ is the velocity, normalized so that $\|v\| = 1$.

Note: now momentum is no longer mass times velocity!

The force is

$$F_i = \frac{\partial L}{\partial q^i} = \frac{\partial}{\partial q^i}(eA_j\dot{q}^j) = e\frac{\partial A_j}{\partial q^i}\dot{q}^j$$

So E-L eqns say

$$\dot{p} = F$$

really $A_i(q(t))$

$$\frac{d}{dt}(mv_i + eA_i) = e\frac{\partial A_j}{\partial q^i}\dot{q}^j$$

$$m\frac{dv_i}{dt} = e\frac{\partial A_j}{\partial q^i}\dot{q}^j - e\frac{d}{dt}A_i$$

$$m\frac{dv_i}{dt} = e\frac{\partial A_j}{\partial q^i}\dot{q}^j - e\frac{\partial A_i}{\partial q^j}\dot{q}^j$$

$$= e \underbrace{\left(\frac{\partial A_j}{\partial q^i} - \frac{\partial A_i}{\partial q^j} \right)}_{F_{ij}} \dot{q}^j$$

F_{ij} — the electromagnetic field $F = dA$.

So we get:

$$\boxed{m \frac{dv_i}{dt} = e F_{ij} \dot{q}^j}$$

the Lorentz force law

(usually called the "Lorenz" force law)

We've looked at a point particle & tried

$$S = m \cdot (\text{arclength}) + \int A$$

or proper time

A 1-form can be integrated over a 1-dimensional path.

In string theory we boost the dimension by 1 & consider a string tracing out a 2d surface a time passes

The action is the obvious modification of that for a point particle:

$$S = m \cdot \text{area} \qquad + \qquad e\int B$$

("string tension": mass/length

(can integrate a 2-form B over the 2d surface traced out by the string. B is the "Kalb-Ramond field".

An Alternative Lagrangian:

This Lagrangian for a charged particle:

$$L = m\|\dot{q}\| + eA_i\dot{q}^i$$

has reparameterization symmetry, just as for the uncharged particle, but there's another Lagrangian we can use, that doesn't:

$$L = \tfrac{1}{2}m\dot{q}\cdot\dot{q} + eA_i\dot{q}^i$$

Nice features of this:

- looks formally like "$\frac{1}{2}mv^2$" in nonrelativistic mechanics
- now there's no square root, so it's everywhere differentiable (and no trouble with ~~parameterizing~~ paths in timelike vs. spacelike directions)

What E-L eqns does this Lagrangian give?

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m\dot{q}_i + e A_i$$

Very similar to before!

$$F_i = \frac{\partial L}{\partial q^i} = e \frac{\partial A_j}{\partial q^i} \dot{q}^j$$

So we get

$$\frac{d}{dt}(m\dot{q}_i + e A_i) = e \frac{\partial A_j}{\partial q^i} \dot{q}^j$$

$$m\ddot{q}_i = e F_{ij} \dot{q}^j$$

— almost as before. The only difference is we have $m\ddot{q}_i$ instead of $m\dot{v}_i$ where $v_i = \frac{\dot{q}_i}{\|\dot{q}\|}$. So the old E-L eqns of motion reduce to the new ones if we pick a parameterization with $\|\dot{q}\| = 1$ — parameterization by proper time.

20 April 2005

Starting from the Lagrangian for a relativistic charged particle in an electromagnetic field

$$L = \frac{1}{2} m\dot{q}_i\dot{q}^i + eA_i\dot{q}^i$$

let's work out the Hamiltonian. Recall that for our reparameterization-invariant Lagrangian

$$L = m\sqrt{\dot{q}_i\dot{q}^i} + eA_i\dot{q}^i$$

we got $H=0$: time translation was a gauge symmetry. Now it's not!

$$H = p_i\dot{q}^i - L$$

and now

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m\dot{q}_i + eA_i$$

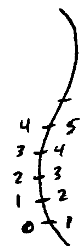
so

$$\begin{aligned} H &= (m\dot{q}_i + eA_i)\dot{q}^i - \left(\tfrac{1}{2}m\dot{q}_i\dot{q}^i + eA_i\dot{q}^i\right) \\ &= \tfrac{1}{2}m\dot{q}_i\dot{q}^i \end{aligned}$$

(vaguely like how a nonrelativistic particle in a potential V has $H = p_i\dot{q}^i - L = 2K - (K-V) = K+V$, but now the "potential" $V = eA_i\dot{q}^i$ is linear in velocity so now $H = p_i\dot{q}^i - L = (2K-V)-(K-V) = K$)

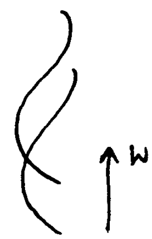
Now H is not zero, & the fact that it's conserved says $\|\dot{q}(t)\|$ is constant as a function of t, so the particle's path is parameterized by proper time up to rescaling of t. I.e. we're getting "conservation of 'speed'" rather than some more familiar "conservation of energy." The reason is, this Hamiltonian comes from the symmetry

$$q_s(t) = q(t+s)$$



instead of spacetime translation symmetry

$$q_s(t) = q(t) + sw \qquad w \in \mathbb{R}^{n+1}$$



Our Lagrangian

$$L(q,\dot{q}) = \tfrac{1}{2} m \|\dot{q}\|^2 + A_i(q)\dot{q}^i$$

has time translation symmetry iff A is translation invariant. In general, then, there's no conserved "energy" for our particle corr. to translations in time.

The General Relativistic Particle

In GR, spacetime Q is an $(n+1)$-dimensional Lorentzian manifold, namely a smooth $(n+1)$-dimensional manifold with a Lorentzian metric g, namely:

1) For each $x \in Q$, we have a bilinear map

$$g(x) : T_x Q \times T_x Q \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \cancel{g(x)(v,w)}$$

(or $g(v,w)$ for short)

2) ~~In some~~ With respect to some basis of $T_x Q$ we have

$$g(v,w) = g_{ij} v^i w^j$$

where

$$g_{ij} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & \ddots \\ & & & & -1 \end{pmatrix}$$

(Of course we can write $g(v,w) = g_{ij} v^i w^j$ in any basis, but for most bases g_{ij} will have a different form)

3) $g(x)$ varies smoothly with x

The Lagrangian for a free point particle in the spacetime Q is

$$L(q,\dot{q}) = m\sqrt{g(q)(\dot{q},\dot{q})}$$
$$= m\sqrt{g_{ij}\dot{q}^i\dot{q}^j}$$

— just like special relativity but with η_{ij} replaced by g_{ij}. Or, we could use

$$L(q,\dot{q}) = \tfrac{1}{2}mg(q)(\dot{q},\dot{q})$$
$$= \tfrac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j$$

The big difference is that now spacetime translation symmetry (& rotation, boost symmetry) is gone! So: no conserved energy-momentum (angular momentum, velocity of center of energy) anymore!

22 April 2005

Suppose Q is a Lorentzian manifold with metric g &

$$L: TQ \to \mathbb{R}$$

is the Lagrangian of a free particle:

$$L(q,\dot{q}) = \tfrac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j$$

Let's figure out the Euler-Lagrange Equations:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = mg_{ij}\dot{q}^j$$

Note: velocity $\dot{q}$ is a tangent vector, momentum p is a cotangent vector, and we need the metric to relate them:

$$g : T_qM \times T_qM \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto g(v,w)$$

gives

$$T_qM \longrightarrow T_q^*M$$
$$v \longmapsto g(v,-)$$

or in coordinates, the tangent vector v^i gets mapped to the cotangent vector $g_{ij}v^j$. This is lurking behind the passage from $\dot{q}^i$ to the momentum $mg_{ij}v^j$.

Back to business:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = mg_{ij}\dot{q}^j$$

$$F_i = \frac{\partial L}{\partial q^i} = \frac{\partial}{\partial q^i}\left(\frac{1}{2}mg_{jk}(q)\dot{q}^j\dot{q}^k\right)$$
$$= \frac{1}{2}m\,\partial_i g_{jk}\dot{q}^j\dot{q}^k \qquad \left(\partial_i := \frac{\partial}{\partial q^i}\right)$$

So, E-L eqs say

$$\frac{d}{dt}\, mg_{ij}\dot{q}^j = \frac{1}{2}m\,\partial_i g_{jk}\dot{q}^j\dot{q}^k$$

Note: the motion is independent of the mass.

We can rewrite this geodesic equation as follows

$$\frac{d}{dt} m g_{ij} \dot{q}^j = \frac{1}{2} m \partial_i g_{jk} \dot{q}^j \dot{q}^k$$

note $g_{ij} = g_{ij}(q(t))$

$$\partial_k g_{ij} \dot{q}^k \dot{q}^j + g_{ij} \ddot{q}^j = \frac{1}{2} \partial_i g_{jk} \dot{q}^j \dot{q}^k$$

$$g_{ij} \ddot{q}^j = \left(\frac{1}{2} \partial_i g_{jk} - \partial_k g_{ij} \right) \dot{q}^j \dot{q}^k$$

$$= \underbrace{\frac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} - \partial_j g_{ik} \right)}_{-\Gamma_{ijk}} \dot{q}^j \dot{q}^k$$

symmetry of the metric

$$\ddot{q}_i = g_{ij} \ddot{q}^j = -\Gamma_{ijk} \dot{q}^j \dot{q}^k$$

$$\ddot{q}^i = -\Gamma^i_{jk} \dot{q}^j \dot{q}^k$$

So we see that $\ddot{q}$ can be computed in terms of $\dot{q}$ & the Christoffel symbols Γ^i_{jk}, which are really the connection that a Lorentzian manifold has (the Levi Civita connection) — i.e. the rule for "parallel transporting" tangent vectors.

We could also consider a particle of charge e on our Lorentzian manifold, in an electromag. field with vector potential A:

$$L = \frac{1}{2} m g_{ij} \dot{q}^j \dot{q}^k + e A_i \dot{q}^i$$

Not surprisingly this gives E-L eqns:

$$m \ddot{q}_i = -m \Gamma_{ijk} \dot{q}^j \dot{q}^k + e F_{ij} \dot{q}^j$$