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From the Lagrangian to the Hamiltonian Approach (cont)

Given $L: TQ \to \mathbb{R}$, we now know a coordinate-free way of describing the map

$$\lambda: TQ \longrightarrow T^*Q$$
$$(q,\dot{q}) \longmapsto (q,p)$$

given in local coordinates by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

We say L is <u>regular</u> if λ is a diffeomorphism from TQ to some open subset $X \subseteq T^*Q$. In this case we can describe what our system is doing equally well by specifying position & velocity:

$$(q,\dot{q}) \in TQ$$

or position & momentum

$$(q,p) = \lambda(q,\dot{q}) \in X.$$

We call X the <u>phase space</u> of the system. In practice $X = T^*Q$, & then L is said to be <u>strongly regular</u>.

<u>Examples</u>: A particle in a Riemannian manifold (Q,g) in a potential $V: Q \to \mathbb{R}$ has Lagrangian

$$L(q,\dot{q}) = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j - V(q)$$

Here
$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij} \dot{q}^j$$

So

$$\lambda(q,\dot q) = (q,\ mg(\dot q,-))$$

so L is strongly regular since

$$\begin{aligned} T_qQ &\longrightarrow T_q^*Q \\ v &\longmapsto g(v,-) \end{aligned}$$

is 1-1 & onto, i.e. the metric is nondegenerate.

<u>Example:</u> A general relativistic particle with charge e in an electromagnetic vector potential A has Lagrangian

$$L(q,\dot q) = \tfrac{1}{2} m g_{ij}\dot q^i \dot q^j - eA_i \dot q^i$$

& thus

$$p_i = \frac{\partial L}{\partial \dot q^i} = m g_{ij}\dot q^j + eA_i$$

This L is still strongly regular, but now each map

$$\begin{aligned} \lambda\big|_{T_qQ} : T_qQ &\longrightarrow T_q^*Q \\ \dot q &\longmapsto mg(\dot q,-) + eA(q) \end{aligned}$$

is affine rather than linear.

<u>Example:</u> The free general relativistic particle with reparameterization-invariant Lagrangian:

$$L(q,\dot q) = m\sqrt{g_{ij}\dot q^i \dot q^j}$$

This is terrible from the perspective of regularity properties — not differentiable when $g_{ij}\dot q^i\dot q^j$ vanishes, & undefined when this quantity is negative! Where it's defined,

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{m g_{ij} \dot{q}^j}{\|\dot{q}\|}$$

(where $\dot{q}$ is timelike), we can ask about regularity. Alas, the map λ is not 1-1 where defined since multiplying $\dot{q}$ by some number has <u>no effect</u> on p! (This is related to the reparameterization invariance — this always happens with reparameterization-inv't. Lagrangians)

<u>Example</u>: Here's a Lagrangian that's regular but not strongly regular. Let $Q = \mathbb{R}$ &

$$L(q,\dot{q}) = f(\dot{q})$$

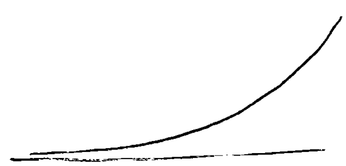
so that

$$p = \frac{\partial L}{\partial \dot{q}} = f'(\dot{q})$$

This will be regular but not strongly so if $f' : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism from $\mathbb{R}$ to some proper subset $U \subset \mathbb{R}$. For example, take $f(\dot{q}) = e^{\dot{q}}$ so $f' : \mathbb{R} \xrightarrow{\sim} (0,\infty) \subset \mathbb{R}$.

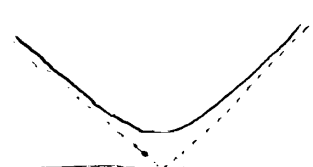
So

$$L(q,\dot{q}) = e^{\dot{q}}$$



(slope is positive)

or $$L(q,\dot{q}) = \sqrt{1+\dot{q}^2}$$



(slope is between -1 & 1)

etc.

Now let's assume L is regular, so

$$\lambda : TQ \xrightarrow{\sim} X \subseteq T^*Q$$
$$(q, \dot{q}) \longmapsto (q, p)$$

This lets us have the best of both worlds: we can identify TQ with X using λ. This lets us treat q^i, p^i, L, H, etc. all as functions on X (or TQ), thus writing

$$\dot{q}^i \qquad (\text{fn on } TQ)$$

for the function

$$\dot{q}^i \circ \lambda^{-1} \qquad (\text{fn on } X)$$

In particular

$$\dot{p}_i := \frac{\partial L}{\partial q^i} \qquad (\text{Euler-Lagrange eqn.})$$

which is really a fn on TQ, will be treated as a fn on X.

Now, let's calculate:

$$\begin{aligned} dL &= \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i \\ &= \dot{p}_i \, dq^i + p_i \, d\dot{q}^i \end{aligned}$$

while

$$\begin{aligned} dH &= d(p_i \dot{q}^i - L) \\ &= \dot{q}^i dp_i + p_i d\dot{q}^i - dL \\ &= \dot{q}^i dp_i + p_i d\dot{q}^i - (\dot{p}_i \, dq^i + p_i d\dot{q}^i) \\ &= \dot{q}^i dp_i - \dot{p}_i dq^i \end{aligned}$$

$$\boxed{dH = \dot{q}^i dp_i - \dot{p}^i dq_i}$$

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Assume the Lagrangian $L: TQ \to R$ is regular, so

$$\lambda : TQ \xrightarrow{\sim} X \subseteq T^*Q$$
$$(q,\dot{q}) \mapsto (q,p)$$

is a diffeomorphism. This lets us regard both L and the Hamiltonian $H = p_i \dot{q}^i - L$ as functions on the phase space X, and use $(q^i, \dot{q}^i)$ as local coordinates on X. As we saw last time, this gives us

$$dL = \dot{p}_i dq^i + p_i d\dot{q}^i$$
$$dH = \dot{q}^i dp_i - \dot{p}_i dq^i .$$

But we can also work out dH directly, this time using local coordinates (q^i, p_i), to get

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i .$$

Since dp_i, dq^i form a basis of 1-forms, we conclude:

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}}$$

HAMILTON'S EQUATIONS

Though $\dot{q}^i$ and $\dot{p}_i$ are just functions of X, when the E-L equations hold for some path $q:[t_0,t_1]\longrightarrow Q$, they will be the time derivatives of q^i and p_i. So when the E-L equations hold, Hamilton's equations describe the motion of a point $x(t)=(q(t),p(t))\in X$. In fact, Hamilton's equations are just the Euler-Lagrange equations in disguise: the equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

really just lets us recover the velocity $\dot{q}$ as a function of q & p, inverting the formula

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

which gave p as a fn. of q & $\dot{q}$. So we get a formula for

$$\begin{aligned} \lambda^{-1} : X &\longrightarrow TQ \\ (q,p) &\longmapsto (q,\dot{q}) \end{aligned}$$

Given this, the other Hamilton eqn

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

is secretly the E-L eqn

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \text{or} \quad \dot{p}_i = \frac{\partial L}{\partial q^i}$$

These are the same because $\frac{\partial H}{\partial q^i} = \frac{\partial}{\partial q^i}(p_i\dot{q}^i - L) = -\frac{\partial L}{\partial q^i}$.

Example: A particle in $Q=\mathbb{R}^n$ in a potential $V:\mathbb{R}^n \to \mathbb{R}$.

This has Lagrangian $L(q,\dot{q}) = \frac{m}{2}\|\dot{q}\|^2 - V(q)$, which gives

$$p = m\dot{q} \qquad \text{so} \qquad \dot{q} = \frac{p}{m} \qquad \left(\text{really: } \dot{q}^i = \frac{g^{ij}p_j}{m}\right)$$

and Hamiltonian

$$H(q,p) = p_i\dot{q}^i - L = \frac{1}{m}\|p\|^2 - \left(\frac{\|p\|^2}{2m} - V\right)$$
$$= \frac{1}{2m}\|p\|^2 + V(q).$$

So Hamilton's equations say

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \implies \dot{q} = \frac{p}{m}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \implies \dot{p} = -\nabla V$$

The first just recovers $\dot{q}$ as a function of p; the second is $F = ma$.

Hamilton's eqns push us toward the viewpoint where p & q have equal status as coordinates on the phase space X. Soon, we'll drop the requirement that $X \subseteq T^*Q$ where Q is a configuration space. X will just be a manifold equipped with enough structure to write down Hamilton's eqns starting from any $H: X \to \mathbb{R}$.

The coordinate-free description of this structure is the major 20th century contribution to mechanics: a symplectic structure.

Hamilton's equations from the Principle of Least Action

Before, we obtained the E-L eqns by associating an "action" S to any $q: [t_0, t_1] \longrightarrow Q$ and setting $\delta S = 0$. Now let's get Hamilton's eqns directly by assigning an action S to any path $x: [t_0, t_1] \longrightarrow X$ and setting $\delta S = 0$. Note: we don't impose any relation between p & $q, \dot{q}$! The relation will follow from $\delta S = 0$.

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Let $\mathcal{P}$ be the space of paths in the phase space X and define the action

$$S : \mathcal{P} \longrightarrow \mathbb{R}$$

by

$$S(x) = \int_{t_0}^{t_1} (p_i \dot{q}^i - H)\, dt$$

where $p_i \dot{q}^i - H = L$. More precisely, write our path x as $x(t) = (q(t), p(t))$ and let

$$S(x) = \int_{t_0}^{t_1} \left(p_i(t) \frac{d}{dt} q^i(t) - H(q(t), p(t)) \right) dt$$

(we write $\frac{d}{dt} q^i$ instead of $\dot{q}^i$ to emphasize that we mean the time derivative rather than a coordinate in phase space.)

Let's show $\delta S = 0 \iff$ Hamilton's equations:

$$
\begin{aligned}
\delta S &= \delta \int (p_i \dot{q}^i - H)\, dt \\
&= \int (\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \delta H)\, dt \\
&\quad \text{Int. by parts} \\
&= \int (\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \delta H)\, dt \\
&= \int \left(\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i} \delta q^i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt \\
&= \int \left(\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) + \delta q^i \left(-\dot{p}_i - \frac{\partial H}{\partial q^i} \right) \right) dt
\end{aligned}
$$

This vanishes $\forall\ \delta x = (\delta q, \delta p)$ if and only if Hamilton's equations:

$$
\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad p_i = -\frac{\partial H}{\partial q^i}
$$

hold.

We've seen two "principles of least action":

1) for paths in configuration space Q, $\delta S = 0 \Rightarrow$ E-L eqns
2) for paths in phase space X, $\delta S = 0 \Rightarrow$ Hamilton's eqns

Additionally, since $X \subseteq T^*Q$, we might consider a third version based on paths in position-velocity space TQ. But when our

Lagrangian is regular, we have a diffeomorphism $\lambda: TQ \xrightarrow{\sim} X$, so this third principle of least action is just a reformulation of (2). However, the really interesting principle of least action involves paths in the extended phase space where we have an additional coordinate for time: $X \times \mathbb{R}$.

Recall the action

$$
\begin{aligned}
S(x) &= \int (p_i \dot{q}^i - H)\, dt \\
&= \int p_i \frac{dq^i}{dt} dt - H dt \\
&= \int p_i dq^i - H dt
\end{aligned}
$$

We can interpret the integrand as a 1-form

$$\beta = p_i dq^i - H dt$$

on $X \times \mathbb{R}$, which has coordinates p_i, q^i, t. So any path

$$x : [t_0, t_1] \longrightarrow X$$

gives a path

$$
\begin{aligned}
c : [t_0, t_1] &\longrightarrow X \times \mathbb{R} \\
t &\longmapsto (x(t), t)
\end{aligned}
$$

and the action becomes the integral of a 1-form over a curve:

$$S(x) = \int p_i dq^i - H dt = \int_c \beta$$