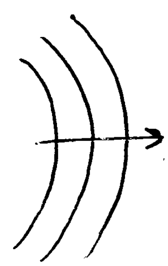
Waves versus Particles — The Hamilton Jacobi Equation

In quantum mechanics we discover that every particle — electrons, photons, etc. — is a wave and vice versa. Already Newton had a particle theory of light ("corpuscules") & various physicists argued against it by pointing out that diffraction is best explained by a wave theory. We've talked about geometrical optics, an approximation in which light consists of particles moving along geodesics. Here we start with a Riemannian manifold (Q,g) as space, but use the new metric

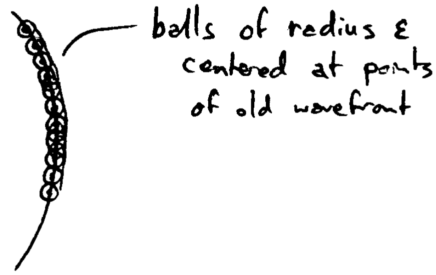
$$h_{ij} = n^2 g_{ij}$$

where $n\colon Q \to (0,\infty)$ is the index of refraction.

Huygens considered this same setup (in simpler language) & considered the motion of a <u>wavefront</u>:



and saw that the wavefront is the envelope of a bunch of little wavefronts centered at points of the big wavefront:



In short, the wavefront moves at unit speed in the normal direction, with respect to the "optical metric" h. We can think about the distance function

$$d: Q \times Q \longrightarrow [0, \infty)$$

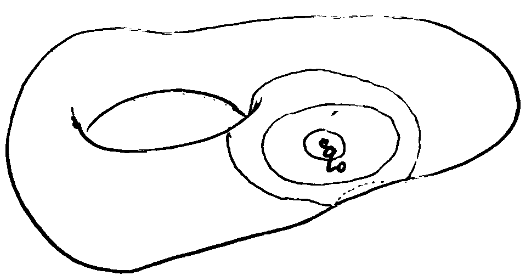
on the Riemannian manifold (Q,h) – where

$$d(q_0, q_1) = \inf_{\wp} (\text{arclength})$$

where $\wp = \{\text{paths from } q_0 \text{ to } q_1\}$. (Secretly this $d(q_0,q_1)$ is the least action – the infimum of action over all paths from q_0 to q_1.) Using this we get the wavefronts centered at $q_0 \in Q$ as the level sets

$$\{ q : d(q_0, q) = c \}$$

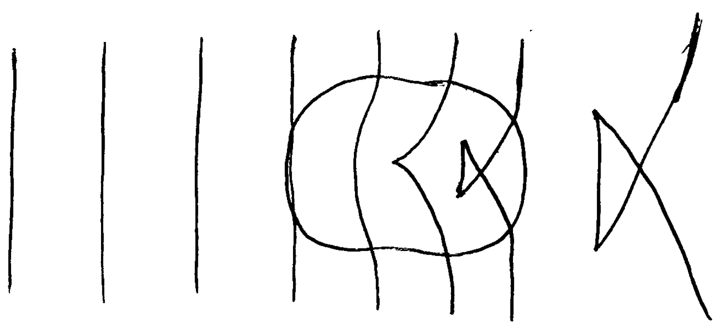
– at least for small $c > 0$.



For larger c the level sets can cease to be smooth – we say a <u>catastrophe</u> occurs – and then the wavefronts are no longer the level sets.

This can happen for topological reasons:

It can also happen for geometrical reasons:



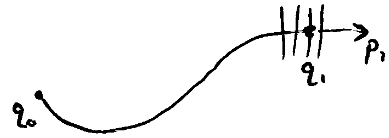
Assuming no such catastrophes occur, we can approximate the waves of light by a wavefunction:

$$\psi(q) = A(q)e^{ikd(q,q_0)}$$

where k is the wavenumber of the light (i.e. its color) and $A: Q \to \mathbb{R}$ describes the amplitude of the wave, which drops off far from q_0. This becomes the <u>eikonal approximation</u> in optics once we figure out what A should be.

Hamilton and Jacobi focused on distance $d: Q \times Q \to [0,\infty)$ as a function of <u>2</u> variables & called it W, <u>Hamilton's principal function</u>. They noticed:

$$\frac{\partial}{\partial q_1^i} W(q_0,q_1) = p_{1i}$$



where p_1 is a cotangent vector "pointing normal to the wavefronts."

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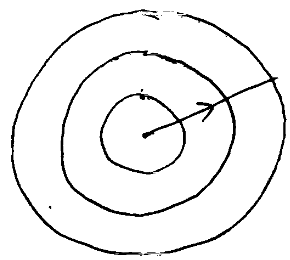
The Hamilton-Jacobi equations

We've seen that in optics, particles of light move along geodesics, but wavefronts are level sets of the distance functions



at least while the level sets remain smooth. In the eikonal approximation, light is described by waves

$$\psi : Q \longrightarrow \mathbb{C}$$
$$\psi(q_1) = A(q_1) e^{ikW(q_0, q_1)}$$

where (Q, h) is a Riemannian manifold, h is the optical metric, $q_0 \in Q$ is the light source, k is the frequency &

$$W : Q \times Q \longrightarrow [0, \infty)$$

is the distance function on Q, or <u>Hamilton's principal function</u>:

$$W(q_0, q_1) = \inf_{q \in \mathcal{P}} S(q)$$

where $\mathcal{P}$ is the space of paths from q_0 to q & $S(q)$ is the action of the path q, i.e. its arclength. This is begging to be generalized to other Lagrangian systems! (at least in retrospect!)

We also saw that

$$\frac{\partial}{\partial q_1^i} W(q_0, q_1) = p_{1i}$$

"points normal to the wavefront" — really the tangent vector

$$p_1^i = h^{ij} p_{1j}$$

points in this direction. In fact $k p_{1i}$ is the <u>momentum</u> of the light passing through q_1. This foreshadows quantum mechanics! (Note: in QM, the momentum is a derivative operator — we get p by differentiating the wavefunction!)

Jacobi generalized this to the motion of point particles in a potential $V : Q \to \mathbb{R}$, using the fact that a particle of energy E traces out geodesics in the metric

$$h_{ij} = \frac{2(E-V)}{m} g_{ij}$$

We've seen this reduces point particle mechanics to optics — but only for particles of fixed energy $\underline{E}$. Hamilton went further, & we now can go further still.

Suppose Q is any manifold & $L : TQ \to \mathbb{R}$ is any function (Lagrangian). Define Hamilton's principal function

$$W : Q \times \mathbb{R} \times Q \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

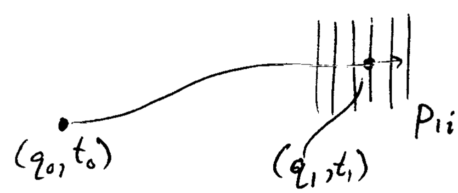
$$W(q_0, t_0; q_1, t_1) = \inf_{q \in \mathcal{P}} S(q)$$

where
$$\mathcal{P} = \{ q : [t_0, t_1] \to Q : \ q(t_0) = q_0 \ \& \ q(t_1) = q_1 \}$$

and
$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, dt$$

W is just the least action for a path from (q_0, t_0) to (q_1, t_1); it'll be smooth if (q_0, t_0) & (q_1, t_1) are close enough — so let's assume that. In fact, we have

$$\frac{\partial W}{\partial q_1^i} = p_{1i}$$



where p_1 is the momentum of the particle going from q_0 to q_1, at time t_1, and

$$\frac{\partial W}{\partial q_0^i} = -p_{0i} \qquad \text{- momentum at time } t_0$$

$$\frac{\partial W}{\partial t_1} = -H_1 \qquad \text{- energy at time } t_1$$

$$\frac{\partial W}{\partial t_0} = H_0 \qquad \text{+ energy at time } t_0$$

really $H_1 = H_0$ — energy is conserv

These are the Hamilton Jacobi equations. The mysterious minus sign in front of energy was seen before in the 1-form

$$\beta = p_i dq^i - H dt$$

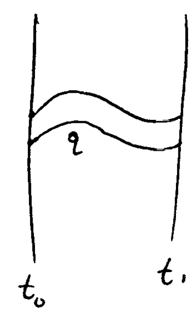
on the extended phase space $X \times \mathbb{R}$. Maybe the best way to get the Hamilton-Jacobi eqns. is from this extended phase space formulation. But for now...

Given (q_0, t_0), (q_1, t_1), let

$$q : [t_0, t_1] \longrightarrow Q$$

be the action-minimizing path from q_0 to q_1. Then

$$W(q_0, t_0 ; q_1, t_1) = S(q)$$



Now vary q_0 & q_1 a bit & thus vary the action-minimizing path, getting a variation δq which does <u>not</u> vanish at t_0 & t_1. We get

$$\begin{aligned}
\delta W &= \delta S \\
&= \delta \int_{t_0}^{t_1} L(q, \dot{q})\, dt \\
&= \int_{t_0}^{t_1} \frac{\partial L}{\partial q^i} \delta q^i + \underbrace{\frac{\partial L}{\partial \dot{q}^i}}_{p_i} \delta \dot{q}^i \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \delta q^i - \dot{p}_i \delta q^i \right) dt + p_i \delta q^i \Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \dot{p}_i \right) \delta q^i \, dt
\end{aligned}$$

zero: E-L eqns. hold since q minimizes the action.

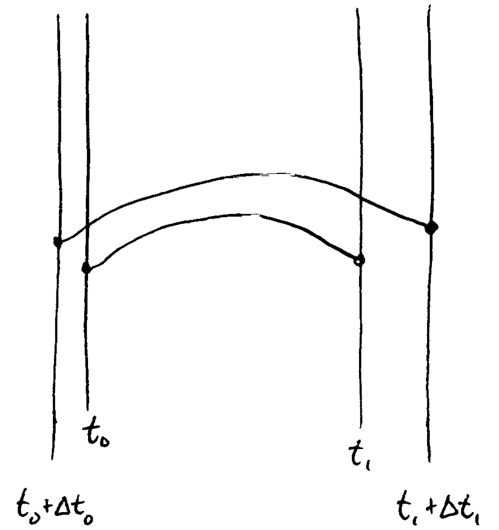
so

$$\delta W = p_{1i}\,\delta q_1^i - p_{0i}\,\delta q_0^i$$

so

$$\frac{\partial W}{\partial q_1^i} = p_{1i} \quad \& \quad \frac{\partial W}{\partial q_0^i} = -p_{0i}$$

These are 2 of the 4 Hamilton-Jacobi eqns! To get the other two, we need to vary t_0 & t_1:



Now change in W will involve Δt_0, Δt_1.

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We want to derive the Hamilton-Jacobi equations describing the derivatives of Hamilton's principal function

$$W(q_0,t_0\,;\,q_1,t_1) = \inf_{q\in\mathcal{P}} S(q).$$

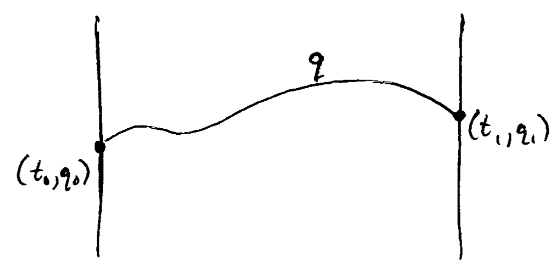
where $\mathcal{P}$ is the space of paths $q:[t_0,t_1]\to Q$ with $q(t_0)=q_0$, $q(t_1)=q_1$, &

$$S(q) = \int_{t_0}^{t_1} L(q,\dot q)\,dt$$

where the Lagrangian $L: TQ \to \mathbb{R}$ will now be assumed <u>regular</u> so that

$$\lambda: TQ \longrightarrow X \subseteq T^*Q$$
$$(q,\dot q) \longmapsto (q,p)$$

is a diffeomorphism. We need to assume that (q_0,t_0) is close enough to (q_1,t_1) that there is a unique $q \in \mathcal{P}$ that minimizes the action S, and assume that this q depends smoothly on $u = (q_0,t_0;q_1,t_1) \in (Q\times\mathbb{R})^2$. We'll think of q as a function of u:



$$(Q\times\mathbb{R})^2 \longrightarrow \mathcal{P}$$
$$u \longmapsto q$$

defined only when (q_0,t_0) & (q_1,t_1) are sufficiently close.

Then Hamilton's principal function is

$$\begin{aligned} W(u) := W(q_0,t_0;q_1,t_1) &= S(q) \\ &= \int_{t_0}^{t_1} L(q,\dot q)\,dt \\ &= \int_{t_0}^{t_1} (p\dot q - H(q,p))\,dt \\ &= \int_{t_0}^{t_1} \underbrace{p\,dq - H\,dt}_{\beta} \\ &= \int_C \beta \end{aligned}$$

where $\beta = p\,dq - H(q,p)\,dt$ is a 1-form on the <u>extended phase space</u> $X \times \mathbb{R}$, and C is a curve in the extended phase space:

$$C(t) = (q(t), p(t), t) \in X \times \mathbb{R}$$

Note C depends on the curve $q \in \mathcal{P}$, which depends on $u = (q_0, t_0; q_1, t_1) \in (Q \times \mathbb{R})^2$.

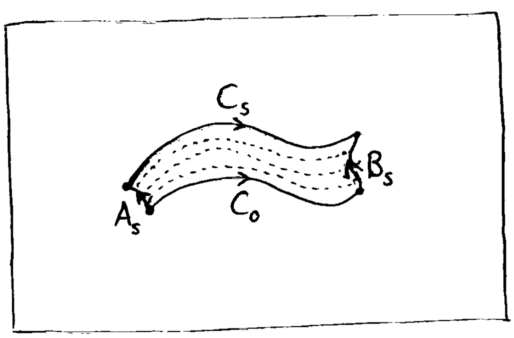
So now let's differentiate

$$W(u) = \int_C \beta$$

with respect to u & get the Hamilton-Jacobi equations from β. Let u_s be a 1-parameter family of points in $(Q \times \mathbb{R})^2$ & work out

$$\frac{d}{ds} W(u_s) = \frac{d}{ds} \int_{C_s} \beta$$

where C_s depends on u_s as above.



$X \times \mathbb{R}$

Let's compare

$$\int_{C_0} \beta \qquad \text{and} \qquad \int_{A_s + C_s - B_s} \beta = \int_{A_s} \beta + \int_{C_s} \beta + \int_{B_s} \beta$$

Since C_0 minimizes the action among paths with the given endpoints, & the curve $A_s + C_s + B_s$ has the same endpoints, we get:

$$\frac{d}{ds}\int_{A_s + C_s - B_s} \beta = 0$$

(Though $A_s + C_s + B_s$ is not smooth, we can approximate it by one that is....) So

$$\frac{d}{ds}\int_{C_s} \beta = \frac{d}{ds}\int_{B_s} \beta - \frac{d}{ds}\int_{A_s} \beta \qquad \text{at } s=0$$

Note

$$\frac{d}{ds}\int_{A_s} \beta = \frac{d}{ds}\int \beta(A_r')\, dr$$

$$= \beta(A_0')$$

where $A_0' = v$ is the tangent vector of A_s at $s=0$.
Similarly,

$$\frac{d}{ds}\int_{B_s} \beta = \beta(w)$$

where $w = B_0'$. So:

$$\frac{d}{ds} W(u_s) = \beta(w) - \beta(v)$$

where w keeps track of the change of (q_1, p_1, t_1) as we move C_s & v keeps track of (q_0, p_0, t_0). Since $\beta = p^i dq_i - H dt$, we get:

$$\frac{\partial W}{\partial q_1^i} = p_1^i$$

$$\frac{\partial W}{\partial t_1} = -H$$

& similarly:

$$\frac{\partial W}{\partial q_0^i} = -p_0^i$$

$$\frac{\partial W}{\partial t_0} = H$$

So: if we define a wavefunction:

$$\psi(q_0, t_0; q_1, t_1) = e^{iW(q_0, t_0; q_1, t_1)/\hbar}$$

we get

$$\frac{\partial \psi}{\partial t_1} = \frac{-i}{\hbar} H_1 \psi$$

$$\frac{\partial \psi}{\partial q_1^i} = \frac{i}{\hbar} p_1 \psi$$

familiar nowadays from QM!