16 May 2005

Towards Symplectic Geometry

Last time we saw that the 1-form

$$\beta = p_i dq^i - H dt$$

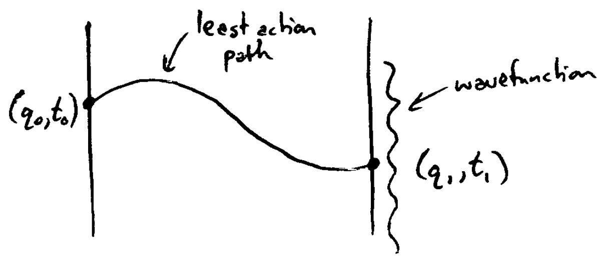
on the extended phase space

$$X \times \mathbb{R} \ni (q, p, t)$$

can be integrated to get the action, with least action giving "Hamilton's principal function" $W(q_0, t_0; q_1, t_1)$. Differentiating this we get <u>momentum</u> & <u>energy</u> – the Hamilton—Jacobi equations. These foreshadow quantum mechanics, since we can define a <u>wavefunction</u>

$$\psi(q_0, t_0; q_1, t_1) = e^{iW(q_0, t_0, q_1, t_1)/\hbar}$$

which (approximately) gives the amplitude for a quantum particle to get from (q_0, t_0) to (q_1, t_1)



In full-fledged quantum mechanics we instead calculate ψ exactly by:

$$\psi(q_0, t_0; q_1, t_1) = \int_{\mathcal{P}} e^{iS(q)} \mathcal{D}q$$

where $\mathcal{P} = \{q : [t_0, t_1] \to Q : q(t_0) = q_0 \ \& \ q(t_1) = q_1\}$ and Dq makes no sense (yet – except in certain special cases, like a particle in a potential on a Riemannian manifold Q). The case when $\psi = e^{iW/\hbar}$ is a good approximation to the full-fledged "path integral," is precisely the case when classical mechanics is approximately right, though really $\psi = A e^{iW/\hbar}$ works better (the eikonal or WKB approximation).

But in classical mechanics, people focus less on the extended phase space $X \times \mathbb{R}$ than on the phase space X. The 1-form β doesn't live on X, but something does. If $x : [t_0, t_1] \to X$ is a path, then we get $C : [t_0, t_1] \to X \times \mathbb{R}$ with $C(t) = (x(t), t)$ and then

$$S = \int_C \beta = \int_C p_i dq^i - H dt$$

& if x satisfies Hamilton's equations, so H is conserved:

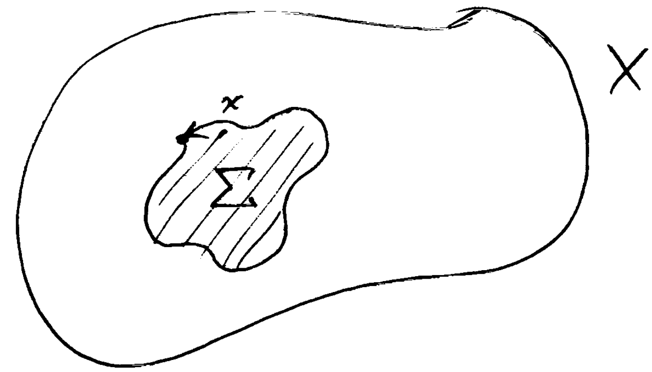
$$S = -E(t_1 - t_0) + \int_x p_i dq^i$$

where E is the energy. So people focus on

$$\alpha = p_i dq^i$$

on X.

If the system executes periodic motion, we can take x to be a loop in X



& then action is

$$S = -E \cdot \text{period} + \int_x \alpha$$

$$= -E \cdot \text{period} + \int_\Sigma d\alpha$$

if Σ is any surface with $\partial\Sigma = x$ by Stokes' thm (this works provided x is contractible) Note

$$d\alpha = dp_i \wedge dq^i$$

is a 2-form on X - people focus on this: the symplectic structure. In the Bohr-Sommerfeld "old quantum mechanics", energy eigenstates correspond to periodic orbits for which $\int_\Sigma dp_i \wedge dq^i = n \cdot 2\pi\hbar = nh$ so that $e^{i\int_\Sigma dp_i \wedge dq^i/\hbar} = 1$.

In the modern "symplectic geometry" approach, people focus on the 1-form α on T^*Q & especially on

$$\omega := d\alpha .$$

Now let's follow suit and develop classical mechanics starting with these.

First, let's describe

$$\alpha = p_i \, dq^i$$

in a coordinate free way. Any tangent vector at $(q,p) \in T^*Q$ is of the form

$$v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$$

using coordinates (q^i, p_i). So

$$\alpha(v) = p_i a^i$$

Alternatively, let

$$\begin{aligned} \pi : T^*Q &\longrightarrow Q \\ (q,p) &\longmapsto q \end{aligned}$$

& we claim

$$\alpha(v) = \underbrace{p}_{\in T_q^*Q} (\underbrace{d\pi(v)}_{\in T_qQ}) \quad \in \mathbb{R}$$

18 May 2005

For any manifold Q (physically the configuration space), T^*Q has a 1-form on it:

$$\alpha = p_i dq^i$$

in local coordinates (q^i, p_i) on T^*Q coming from local coords. on Q. Given any tangent vector $v \in T_{(q,p)}T^*Q$, we can write

$$v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$$

and then

$$\alpha(v) = p_i a^i.$$

α is called the <u>canonical 1-form</u> on T^*Q because it's coordinate-independent. We have

$$\pi : T^*Q \longrightarrow Q$$
$$(q,p) \longmapsto q$$

& thus

$$d\pi : T_{(q,p)}T^*Q \longrightarrow T_qQ$$

& in fact

$$\alpha(v) = p(d\pi(v)) \qquad \forall v \in T_{(q,p)}T^*Q$$

— a coordinate-free description of α. Let's check that this is true. If

$$v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$$

then

$$d\pi(v) = a^i \frac{\partial}{\partial q^i} \quad \in T_q Q$$

where now $\frac{\partial}{\partial q^i}$ are a basis of tangent vectors on Q.

Next

$$p(d\pi(v)) = (p_j \, dq^j)(a^i \frac{\partial}{\partial q^i}) \qquad dq^j(\frac{\partial}{\partial q^i}) = \delta^j_i$$

$$= p_i a^i, \quad \text{which is exactly right.}$$

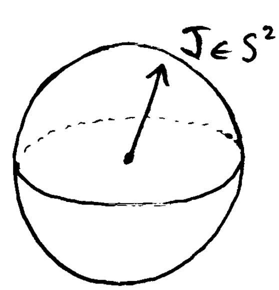
Symplectic geometry focuses not on α but on the 2-form

$$\omega = d\alpha$$

We'll write down Hamilton's equations using ω & show that ω (unlike α) is invariant under the resulting time evolution maps

$$T^*Q \longrightarrow T^*Q \qquad (\text{or } X \to X).$$

If we isolate the key properties of ω that make this work, we can generalize CM to systems where the phase space X is <u>not</u> T^*Q or some open subset of T^*Q. For example, a classical spinning point particle has $X = S^2$:



Such X look <u>locally</u> but not <u>globally</u> like some cotangent bundle, and not in any canonical way.

Here are the key properties of ω:

1) ω is a 2-form on X (in this case $X = T^*Q$)

2) ω is closed: $d\omega = 0$ (in this case exact: $\omega = d\alpha$) cpt phase space $\Rightarrow \omega$ not exact.

3) ω is nondegenerate:

$$\begin{aligned} \theta : T_x X &\longrightarrow T_x^* X \\ v &\longmapsto \omega(v, -) \end{aligned}$$

is an isomorphism (it's 1-1, hence onto)

(in our example if $v = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$ then

$$\begin{aligned} \theta(v) = \omega(v,-) &= dp_j \wedge dq^j \left(a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i} , - \right) \\ &= b_j \, dq^j - a^j dp_j \end{aligned}$$

which is nonzero if v is nonzero:

$$\theta : \frac{\partial}{\partial q^i} \longmapsto -dp_i$$

$$\theta : \frac{\partial}{\partial p_i} \longmapsto dq^i$$

— should remind you of Hamilton's equations!)

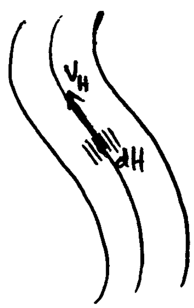
We define a <u>symplectic structure</u> on a manifold X to be a nondegenerate closed 2-form ω on X, & we then call (X, ω) a <u>symplectic manifold</u>.

Now let X be any symplectic manifold (e.g. T^*Q). Given any function $H: X \to \mathbb{R}$ (our <u>Hamiltonian</u>) we can define a vector field v_H on X by:

$$v_H = \theta^{-1}(dH)$$

and then <u>Hamilton's equations</u> say that a "state" $x(t) \in X$ evolves as follows:

$$\frac{d}{dt} x(t) = v_H(x(t))$$



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20 May 2005

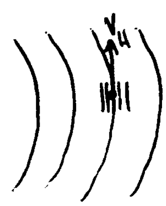
Hamilton's Equations

Let (X, ω) be a symplectic manifold: ω is a nondegenerate closed 2-form, so we get

$$\begin{aligned} \theta: T_x X &\longrightarrow T_x^* X \\ v &\longmapsto \omega(v, -) \end{aligned}$$

an isomorphism. For any $H \in C^\infty(X)$ we thus get the <u>Hamiltonian vector field</u>

$$v_H = \theta^{-1}(dH)$$



level sets of H

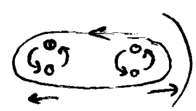
If X is the space of states of a classical system, we describe how a state $x \in X$ evolves in time using the curve $x(t) \in X$ satisfying Hamilton's equations:

$$\frac{d}{dt} x(t) = v_H(x(t))$$

with $x(0) = x$ as its initial conditions. (Note x is being used both for the initial state $x \in X$ and the curve $x : (a,b) \to X$ and $x(0) = x$.) We say v_H is integrable if this differential equation has a (unique) "global solution" $x : \mathbb{R} \to X$ for all $x \in X$. It's easy to find nonintegrable examples:

$$X = T^*\mathbb{R}$$
$$H = \frac{p^2}{2m} - q^4$$

Here the particle shoots off to $\pm\infty$ in finite time.

(There are even solutions in the 5-body Newtonian gravity problem where bodies shoot off to ∞ in finite time! )

Let's see what these Hamilton's equations look like when $X = T^*Q$. Then

$$\omega = dp_i \wedge dq^i$$

and we have

$$\theta : T_x Q \longrightarrow T_x^* Q$$
$$\frac{\partial}{\partial q^i} \longmapsto -dp_i$$
$$\frac{\partial}{\partial p_i} \longmapsto dq^i$$

So
$$\theta^{-1} : T_x^*Q \longrightarrow T_xQ$$
$$dq^i \longmapsto \frac{\partial}{\partial p_i}$$
$$dp_i \longmapsto -\frac{\partial}{\partial q^i}$$

so given $H : T^*Q \longrightarrow \mathbb{R}$ we get

$$\begin{aligned} V_H &= \theta^{-1}(dH) \\ &= \theta^{-1}\left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i\right) \\ &= \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \end{aligned}$$

& thus Hamilton's equations say

$$\frac{d}{dt} x(t) = V_H(x(t))$$

or

$$\frac{d}{dt} q^i(x(t)) = V_H(q^i) = -\frac{\partial H}{\partial p_i}$$

and

$$\frac{d}{dt} p_i(x(t)) = V_H(p_i) = \frac{\partial H}{\partial q^i}$$

Alas, these have a minus sign as compared with the usual Hamilton's eqns.; to cure this we should define

$$\omega = -d\alpha = dq_i \wedge dp^i$$

& get
$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

We've just discovered that any symplectic manifold (M, ω) has a dual or time-reversed version $(M, -\omega)$.

This calculation seems special to the case of a cotangent bundle $X = T^*Q$, but in fact we have:

<u>Darboux's Thm:</u> If (X, ω) is any symplectic manifold and $x \in X$, you can find <u>symplectic coordinates</u> p_i, q^i $(i = 1, \ldots, n)$ in a nbhd. of x s.t.

$$\omega = dq_i \wedge dp^i$$

So:

1) all symplectic manifolds are even-dimensional
2) all symplectic manifolds of the same dimension are locally alike – unlike Riemannian manifolds.

We can think of any $F \in C^\infty(X)$ as an <u>observable</u>: it assigns to any state $x \in X$ a number $F(x) \in \mathbb{R}$ — the result of measuring something about x. We know how states evolve in time:

$$\frac{d}{dt} x(t) = v_H(x(t))$$

How do observables evolve in time? Given an observable $F \in C^\infty(X)$ and a time $t \in \mathbb{R}$ we get a new observable $F_t \in C^\infty(X)$ by

$$F_t(x) = F(x(t))$$

We then have

$$\frac{d}{dt} F_t(x) = \frac{d}{dt} F(x(t))$$

chain rule & Hamilton's eqns.

$$= (V_H F)(x(t))$$

$$= (V_H F_t)(x)$$

$$\frac{d}{dt} F(x(t)) = \nabla F \cdot x'(t)$$
$$= dF(\underbrace{x'(t)}_{v})$$
$$= vF(x(t))$$

so, abstracting:

$$\frac{d}{dt} F_t = V_H F_t$$